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RADIATION ELECTRODYNAMICS OF THE PHOTO-ELECTRON CLOUD PRODUCED BY AN ARBITRARY PHOTON PULSE INCIDENT ON A PLANAR EMITTING SURFACE IN VACUUM

David Dietz

June 1994

Final Report

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13. ABSTRACT *(Maximum 200 words)*

The electromagnetic radiation field produced by the cloud of accelerating electrons induced at a planar photoelectron emitting surface in vacuum by a photon pulse of arbitrary time profile incident upon that surface is derived analytically in the nonrelativistic, small-spot-size regime. This derivation provides scaling relations for the radiated fields explicit in the pulse and surface parameters. Electron cloud dynamics is discussed and integral expressions for the electric and magnetic fields at large but finite (i.e., "finitely-remote")distances from the cloud are derived. The fields are calculated by directly summing contributions over individual electron trajectories, using exact small-spot retarded times, rather than by the usual technique of first forming the electron charge and current densities and integrating them. The resulting integral representation of the finitely-remote fields are correct to first order in v/c and are valid for all time and for somewhat spatially extended charge clouds. These finitely-remote fields are then used to compute asymptotic radiation fields in the limit of the field point going to infinity in a suitable fashion. From these asymptotic radiation quantities in which all the integrations have been fully carried out are then derived. Illustrations of the general results for some sample pulses (constant, linear ramp, triangular, parabolic, and $\sin^2$) are provided.

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I. INTRODUCTION

In this paper we study analytically the electromagnetic radiation field produced by the cloud of accelerating, non-relativistic electrons induced at a planar photoelectron emitting surface in vacuum by a photon pulse of arbitrary time profile and small spatial cross-sectional area incident upon that surface, deriving scaling relations for the radiated fields (hence radiated power, energy, and spectral content) explicit in the pulse and surface parameters.

The *dynamics* of the electron cloud has been of interest in connection with both electromagnetic pulse (EMP) phenomena [1-5] and laser plasma jets [6-7] and has been studied both analytically and numerically. Those studies are concerned with either steady-state dynamics [1,3,6] -- applicable when the electron plasma period is much shorter than the characteristic photon pulse width -- or with transient dynamics [2,4,5,7], applicable when the above condition does not hold. Those authors who obtain complete, strictly analytical results for the transient case [2,7] assume 1-D planar geometry, monoenergetic emission into vacuum, and a "no-screening" approximation [1,2,4,7]. Under these assumptions, they solve Maxwell's equations (simply Gauss's law in this situation) for the electric field *in the space charge region*, coupled to Newton's law for electron motion as well as to a continuity equation for the electron density, to obtain a self-consistent solution to the dynamics problem. This program produces *formal* expressions for the electron density and velocity fields as a function of the single space coordinate and time; these expressions could be used in principle to derive similarly formal expressions for the radiated fields of interest to us here, although this has not been done in the literature as far as we know. However, from our point of view there is a fundamental difficulty with these formal expressions: They can be made to yield expressions for the radiated fields *explicit* in the pulse and surface parameters only in the two special cases of photon pulses which are either constant in time or linearly increasing in time. Indeed, these two special cases are the only ones offered in [2] and [7] as illustrations of their general formalism. The specific difficulty is this. The treatment of the particle dynamics is done from a Lagrangian (in the fluid sense) point of view while the electron density and velocity fields are inherently Eulerian (as is usually the case in the

electrodynamics of extended charge distributions). The formal expressions for the density and velocity fields require a translation from Eulerian to Lagrangian coordinates in order to be expressed explicitly in terms of the pulse and surface parameters ([2], Eqs. (18) and (36); [7], Eqs. (16)-(18)) and this translation can only be done exactly analytically in the two special cases indicated above.

In order then to achieve generality, we abandon the usual technique of using the (Eulerian) charge and current densities to compute the radiation fields; rather, we compute these fields by the novel technique of directly summing contributions to the fields over individual electron trajectories, i.e., we do "Lagrangian (in the fluid sense) electrodynamics". (We point out that this technique is novel only because it is being used as a strictly analytical tool -- the technique of summing contributions over individual particles has been used previously in electromagnetic particle-in-cell (PIC) codes.) This summing process results in integral expressions for the electric and magnetic fields at large but finite (i.e., "finitely-remote") distances. True radiation quantities, however, are obtained only in the limit of the field point going to infinity; we thus demonstrate the existence of this limit and obtain asymptotic radiation quantities in which all the integrations have been fully carried out and which, in addition, are explicit in the pulse and surface parameters. Because we wish to obtain this limit rigorously, the entire paper of necessity takes on a somewhat mathematical flavor in order to support from the outset this major goal. We have, however, relegated most of the mathematical detail -- which involves only elementary techniques -- to four appendicies so that they may be avoided, for the most part, by those readers who find them of lesser interest.

In summary then, we derive expressions for the asymptotic radiation fields based upon a 2-D cylindrical electron cloud of small radius whose dynamics is described by the aforementioned 1-D planar model employing the monoenergetic emission and no-screening approximations. Since in the 1-D model the magnetic field is strictly absent, then that model yields electrostatic interactions between the electrons; hence, application of the 1-D dynamical model to our 2-D electron cloud is tantamount to assuming only electrostatic interactions and so errs in neglecting small magnetic field and retardation effects *in the dynamics*. Such effects are not neglected in our radiation treatment

however; indeed, we require a small spatial cross-sectional area for the photon pulse so that we may adequately represent the retarded time in the radiation integrals.

In Section II we describe the photon pulse more completely. In Section III we discuss electron cloud dynamics, presenting in detail only those features unique to our treatment. In Section IV we derive expressions for the finitely-remote fields; these lead, in Section V, to the asymptotic fields which in turn lead, in Section VI, to all asymptotic radiation quantities of interest. Finally, in Section VII we present illustrations of our general formulation for five specific pulses: constant, linear ramp, triangular, parabolic, and $\sin^2$. Additionally, Appendicies A and B, C, and D contain the mathematical complements to Sections III, IV, and V respectively.

II. PHOTON PULSE DESCRIPTION

The photon pulse, normally incident on the emitting surface, is taken to be one of circular spatial cross-section with spot radius a, and photon frequency ν. This restriction to circular cross-section is not necessary, since we will ignore edge effects in the cylindrical electron cloud, but seems most natural. We assume that photoelectrons are emitted spatially uniformly over the extent of the spot, each with one and the same non-relativistic speed $v_0 > 0$ in a direction normal to the locally planar, smooth emitting surface, with average photoelectron yield (quantum efficiency) over the spot denoted by $Y(\nu)$ (electrons/photon). (Here, "locally" means over a region comparable to that of the spot.) Monoenergetic, normal emission of electrons is certainly not the case physically in general where, even for a normally incident pulse of monochromatic photons, the shape and width of the emitted photoelectron kinetic energy distribution function is highly dependent upon, and varies greatly with, the nature and condition of the emitting surface as well as with the incident photon frequency [8]; but, as we have previously pointed out, monoenergetic (normal) emission is a standard assumption in the literature. We also assume that the incident photon intensity is not high enough to produce a plasma at the surface; and we do not include contributions to the radiation field of induced currents in case the emitting surface is a conductor [6].

The full width of the pulse in time is taken as π/Ω, where $\Omega > 0$ is an angular frequency (sec^{-1}), while the pulse intensity at the emitting surface is given by

$$\hat{I}(t) = Af(t), \qquad 0 \le t \le \pi/\Omega, \tag{2.1}$$

where $A > 0$ is an amplitude (photons/m^2-sec [we use MKSA units throughout]) and f is dimensionless, with $0 \le f(t) \le 1$, $f(t) = 1$ for at least one t, and $f(t) = 0$ for at most $t \in \{0, \pi/\Omega\}$. It is convenient to introduce a dimensionless time variable given by $s = \Omega t$; in terms of s, we define functions I and g by

$$I(s) = \hat{I}(s/\Omega) \geq 0, \qquad g(s) = f(s/\Omega) \geq 0, \qquad 0 \leq s \leq \pi, \tag{2.2}$$

so that

$$I(s) = Ag(s), \qquad 0 \leq s \leq \pi. \tag{2.3}$$

The central construct of our model turns out to be $\int_0^t \hat{I}(t')dt' = A\int_0^t f(t')dt'$ in terms of which all physical quantities of interest may be naturally written. It will often be more convenient to use a dimensionless multiple of this integral: we define

$$G(s) = \int_0^s g(s')ds' \Big/ \int_0^\pi g(s')ds', \qquad 0 \leq s \leq \pi, \tag{2.4}$$

so that

$$G(\Omega t) = \int_0^{\Omega t} g(s)ds \Big/ \int_0^\pi g(s)ds = \int_0^t f(t')dt' \Big/ \int_0^{\pi/\Omega} f(t')dt', \qquad 0 \leq t \leq \pi/\Omega . \tag{2.5}$$

Also, we define

$$N(\Omega) \equiv \int_0^{\pi/\Omega} f(t')dt' = (1/\Omega)\int_0^\pi g(s)ds \tag{2.6}$$

so that

$$\int_0^t f(t')dt' = N(\Omega)G(\Omega t), \qquad 0 \leq t \leq \pi/\Omega; \tag{2.7}$$

further, we exclude as trivial the zero pulse so that $N(\Omega) > 0$. Therefore

$$0 \le G(s) \le 1 \text{ for } 0 \le s \le \pi \text{ and } G(0) = 0,\ G(\pi) = 1. \tag{2.8}$$

We also require that G' exists on $[0, \pi]$ (taking one-sided limits at $0, \pi$); to ensure this we demand that g be continuous on $[0, \pi]$ (or f on $[0, \pi/\Omega]$), and so is properly integrable there, and in that case we have

$$G'(s) = g(s)/\Omega N(\Omega) = I(s)/A\Omega N(\Omega) \ge 0, \qquad 0 \le s \le \pi. \tag{2.9}$$

Hence G is non-decreasing on $[0, \pi]$ and, being continuous there, takes on every value between 0 and 1 (by Eq. (2.8)). In fact, since g vanishes only at most at 0 and π then G' is strictly positive on $(0, \pi)$ and hence G is strictly increasing on $[0, \pi]$.

We point out that we could easily extend our results to improperly integrable pulses g which are continuous only on $(0, \pi)$ but for which $\lim_{s \to 0^+} g(s) = \infty$ or $\lim_{s \to \pi^-} g(s) = \infty$ (or both), so that we could treat, for example, $g(s) = A\,[(\pi - s)^{-1/2} - \pi^{-1/2}]$; and to pulses of infinite width which are continuous on $(0, \infty)$, so that we could treat, for example $g(s) = Ae^{-(s-\bar{s})^2}$, $s \ge 0$ $(\bar{s} > 0)$; but, on physical grounds, we see little reason to do so. We could also allow g to have discontinuities in the interior of its interval of definition so that we could treat, for example, $g(s) = A \cdot \begin{cases} 1/2, & \text{if } 0 \le s < \pi/2 \\ 1, & \text{if } \pi/2 \le s \le \pi \end{cases}$. This last generalization only makes mathematical arguments more cumbersome and provides essentially no additional physical insight. We choose not to pursue any of these possibilities here.

III. ELECTRON CLOUD MOTION

A. Kinematics

At any instant $t \geq 0$, we think of the electron cloud as a circular cylinder of radius a comprised of a continuum of infinitesimally thin, planar, parallel electron disks (or "sheets"), each having been emitted at some instant $t_o \in [0, t]$ from, and each remaining forever parallel to, the planar emitting surface, with precisely one such sheet having been emitted for each $t_o \in [0, t]$. It follows that each sheet may be unambiguously labeled by the time, t_o, at its emission. Observe that the total charge contained in the sheet emitted during infinitesimal time interval δt_o at t_o is (forever)

$$\delta q(t_o) = -eY(v)I(t_o)\pi a^2 \delta t_o = -eY(v)A\Omega N(\Omega)G'(\Omega t_o)\pi a^2 \delta t_o, \tag{3.1}$$

where e is the magnitude of the electron charge, and that this sheet has infinitesimal thickness $\delta z = v_0 \delta t_0$ so that also

$$\delta q(t_o) = -eY\hat{I}(t_o)\pi a^2 \delta z / v_o. \tag{3.2}$$

Further, if we let a z-axis coincide with the symmetry axis of the cylinder and choose as positive the direction of electron emission, then we denote the position at time t of the sheet labeled by t_o as $Z(t_o; t)$ for $(t_o; t) \in [0, \pi/\Omega] \times (0, \infty)$; clearly

$$Z(t_o; t) \geq 0 \text{ for all } t_o, t, \tag{3.3}$$

the emitting surface being located at $z = 0$. Figure 1 depicts the electron cloud and the emitting surface at any particular instant, t.

We assume that the magnitude of the residual positive charge on the emitting surface is precisely the same as the magnitude of the totality of negative charge in the electron cloud; that is, we

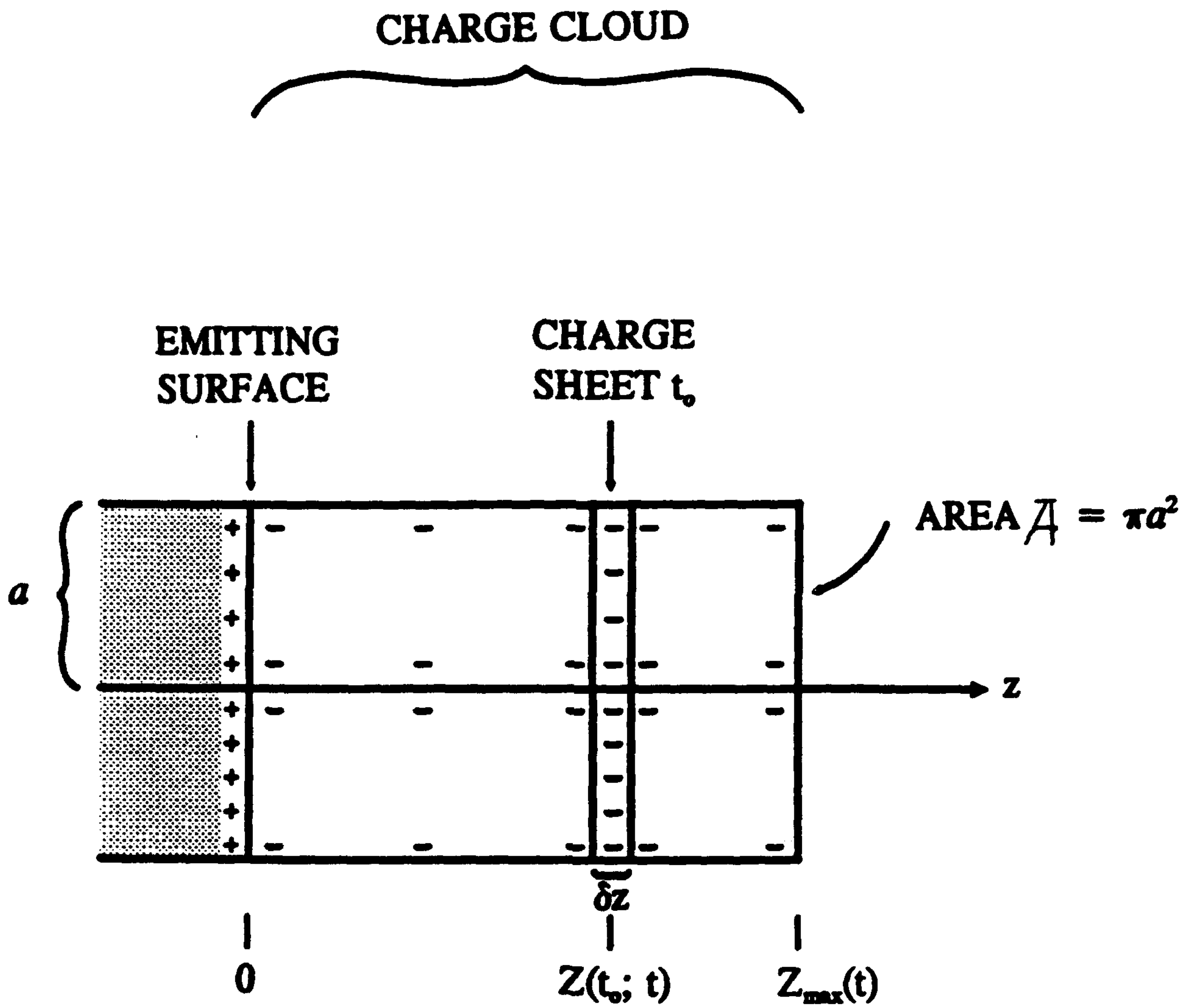


Figure 1. Electron cloud description for dynamics.

assume that neutralization of this residual positive charge by the flow towards it of negative charges originating in the emitting surface -- in case it is metallic -- is negligible during time intervals of interest here. Further, since the thickness of the positively charged layer is on the order of 10^{-9} m [8], we may safely assume that it forms a surface charge distribution.

B. Dynamics

In general, the net electromagnetic force at time t on the charge sheet labeled by t_0 is

$$\mathbf{F}(t_o;t) = [\delta q(t_o)/\Pi\delta z]\,\delta z \iint_{\text{sheet } t_o} \{\mathbf{E}(r,\vartheta,Z(t_o;t),t) + \mathbf{V}(t_o;t) \times \mathbf{B}(r,\vartheta,Z(t_o;t),t)\}dS \tag{3.4}$$

(we assume any one cylindrical coordinate system whose positive z-axis is the one previously specified following Eq. (3.1)) where $\mathbf{V}(t_o;t)$ is the instantaneous sheet velocity, $\mathbf{E}(r, \vartheta, Z(t_o;t), t)$ and $\mathbf{B}(r, \vartheta, Z(t_o;t), t)$ are the instantaneous electric and magnetic fields at the area element dS located at (r, ϑ) on the sheet, and $\Pi = \pi a^2$. Equation (3.4) holds since the sheet has uniform charge density

$$\rho_L(t_o) = \delta q(t_o)/\Pi\,\delta z = (1/\Pi v_o)[\delta q(t_o)/\delta t_o]. \tag{3.5}$$

As alluded to earlier, we approximate the full electromagnetic field $\mathbf{E} + (\mathbf{V} \times \mathbf{B})$ by the net electrostatic field, $\mathbf{E}^{es}$, resulting from the charge in the cloud not on sheet t_0 plus that in the residual surface charge distribution. This net electrostatic field is given by

$$\mathbf{E}^{es}(r,\vartheta,Z(t_o;t),t) = -\mathbf{k}(1/\varepsilon_o)\int_{Z(t_o;t)}^{Z_{max}(t)} d\zeta\,\rho(\zeta,t)\ , \tag{3.6}$$

where $\mathbf{k}$ is the unit vector in the +z direction, $Z_{max}(t)$, $t \geq 0$, is the z-coordinate of the leading edge of the charge cloud, and $\rho(\zeta, t) \leq 0$, $(\zeta, t) \in [0, \infty)^2$, is the z-dependent volume charge density

characterizing the charge cloud; hence we have

$$F(t_o; t) = -\hat{k}[\delta q(t_o)/\varepsilon_o] \int_{Z(t_o;t)}^{Z_{max}(t)} d\zeta \rho(\zeta, t) = \delta q(t_o) \cdot E^{eq}(r, \vartheta, Z(t_o; t), t). \tag{3.7}$$

The reader should note that because the z-component of the electrostatic force at any surface element dS located on charge sheet t_o, due to all the other charge in the system not on sheet t_o, is independent of r (the radial coordinate on the sheet), and because, in addition, all the charge on sheet t_o, other than that belonging to dS, exerts no z-directed force on dS, then the z-component of the force on dS due to all other charges in the system is independent of r; hence, sheet t_o remains forever strictly planar in this approximation. On the other hand, while all the charges not on sheet t_o exert no net radial force on dS in our approximation (Eq. (3.7)), the charges on sheet t_o and not belonging to dS do exert a net radial force on dS and, after some time, significant radial expansion will occur. We assume negligible radial expansion during the time intervals being considered here.

The equation of motion of the center-of-mass of charge sheet t_o is

$$\delta m(t_o)[(\partial^2 Z/\partial t^2)(t_o; t)] = F(t_o; t) \tag{3.8}$$

where $\delta m(t_o)$ is the mass of the sheet and $F(t_o; t)$ given by Eq. (3.7). That is,

$$(\partial^2 Z/\partial t^2)(t_o; t) = (e/m_e \varepsilon_o) \int_{Z(t_o;t)}^{Z_{max}(t)} d\zeta \rho(\zeta, t) \leq 0 \quad . \tag{3.9}$$

Of course, this equation is valid only for those times t for which the sheet t_o has not previously returned to the emitting surface, namely for t satisfying

$$t_o \leq t < t_o^*, \tag{3.10}$$

where t_o^* is given by

$$t_o^* = \sup \{T \in (t_o, \infty) \mid Z(t_o; \tau) > 0 \text{ for all } \tau \in (t_o; T)\} \tag{3.11}$$

(possibly $t_o^* = \infty$); note $t_o^* > t_o$ by Eq. (3.12) below. In other words, given $t_o \in [0, \pi/\Omega]$, to solve the equation of motion we must find a function $t \mapsto Z(t_o; t)$ <u>and</u> a $t_o^* \in (t_o; \infty]$ such that

(1) $Z(t_o; t)$ satisfies Eq. (3.9) for $t \in [t_o, t_o^*)$ and initial conditions

$$Z(t_o; t_o) = 0 \quad \text{and} \quad (\partial Z/\partial t)(t_o; t_o) = v_o > 0 ; \tag{3.12}$$

and

(2) t_o^* and $Z(t_o; t)$ satisfy Eq. (3.11).

Outside the time interval specified by Eq. (3.10) the following must hold:

$$Z(t_o; t) = 0 = (\partial Z/\partial t)(t_o; t) = (\partial^2 Z/\partial t^2)(t_o; t) \text{ if } t \in (-\infty, t_o) \cup (t_o^*, \infty). \tag{3.13}$$

(We will specify values for these functions at $(t_o; t_o^*)$, in case $t_o^* < \infty$, later on.) In particular, it follows from this and Eq. (3.12) that $t \mapsto (\partial Z/\partial t)(t_o; t)$ is discontinuous at $t = t_o$ so that $(\partial^2 Z/\partial t^2)(t_o; t)$ does not exist there; hence the second derivative in Eq. (3.9) should be interpreted to be the appropriate one-sided quantity at $t = t_o$. The physical interpretation of Eq. (3.9) is that the instantaneous acceleration of sheet t_o at time t depends only upon the total amount of charge in the cloud between the sheet itself and the cloud's leading edge, being proportional to that charge.

Equation (3.9), as an equation for unknown $Z(t_o; t)$, is not readily solvable since $Z(t_o; t)$ occurs as a limit of integration and since $Z_{max}(t)$ and $\rho(\zeta, t)$ are also unknown. In order to proceed it is useful to make one additional simplifying assumption, namely, that charge sheets do not pass through one another. This is the so-called "no-screening" or "no-charge-sheet-crossing" (NCSC)

approximation. If such is the case, then the integral in Eq. (3.9) is independent of time and is given by (using Eqs. (3.2), (2.1), and (2.7))

$$\Pi \int_{Z(t_o;t)}^{Z_{max}(t)} d\zeta\, \rho(\zeta, t) = \int_0^{t_o} \delta q(\tau_o) = -eY\Pi \int_0^{t_o} \mathcal{I}(\tau_o) d\tau_o$$
$$= -eYA\Pi \int_0^{t_o} f(\tau_o) d\tau_o = -eYA\Pi N(\Omega) G(\Omega t_o) \qquad \text{(NCSC)}. \tag{3.14}$$

Hence Eq. (3.9) becomes

$$(\partial^2 Z/\partial t^2)(t_o; t) = -[e^2 AYN(\Omega)/m_e\varepsilon_o] G(\Omega t_o) \qquad \text{(NCSC)}, \tag{3.15}$$

valid for $t_o \le t < t_o^*$ (with t_o^* yet to be determined). This is the required equation of motion of charge sheet t_o. It implies that a given sheet t_o experiences a constant (in time) negative acceleration and, furthermore, if $t_o' > t_o$ then sheet t_o' experiences a more negative acceleration than does sheet t_o (since G is strictly increasing on $[0, \pi]$).

We may now integrate Eq. (3.15) twice, subject to initial conditions given by Eq. (3.12), to get

$$(\partial Z/\partial t)(t_o; t) = v_o - [e^2 AYN(\Omega)/m_e\varepsilon_o] G(\Omega t_o)(t - t_o) \tag{3.16}$$

and

$$Z(t_o; t) = v_o(t - t_o) - (1/2)[e^2 AYN(\Omega)/m_e\varepsilon_o] G(\Omega t_o)(t - t_o)^2. \tag{3.17}$$

Defining

$$\omega_p^2 \equiv e^2 (AY/v_o)/m_e\varepsilon_o\ , \tag{3.18}$$

where ω_p is the plasma frequency corresponding to maximal electron emission density AY/v_o, we have

$$(\partial Z/\partial t)(t_o;t) = v_o[1-\omega_p^2 N(\Omega)G(\Omega t_o)(t-t_o)] \tag{3.19}$$

and

$$Z(t_o;t) = v_o(t-t_o)[1-(1/2)\omega_p^2 N(\Omega)G(\Omega t_o)(t-t_o)]\ ; \tag{3.20}$$

also

$$(\partial^2 Z/\partial t^2)(t_o;t) = -v_o\omega_p^2 N(\Omega)G(\Omega t_o)\ . \tag{3.21}$$

We reiterate that these equations are valid only for $t_o \le t < t_o^*$ and for NCSC.

To find t_o^*, we observe that the motion of charge sheet t_o is clearly that of an object rising and falling under "gravitational" acceleration of magnitude $v_o\omega_p^2 N(\Omega)G(\Omega t_o)$ and, as such, $Z(t_o;t)$ increases monotonically from $z = 0$ at $t = t_o$ to

$$z = Z_M(t_o) \equiv (1/2)v_o\ [1/\omega_p^2 N(\Omega)G(\Omega t_o)] = (1/4)v_o T/G(\Omega t_o) \tag{3.22}$$

at

$$t_{1/2}(t_o) \equiv t_o + [1/\omega_p^2 N(\Omega)G(\Omega t_o)] = t_o + (1/2)T/G(\Omega t_o) > t_o \tag{3.23}$$

and then decreases monotonically to $z = 0$ at

$$t_{rm}(t_o) \equiv t_o + T/G(\Omega t_o) > t_o \tag{3.24}$$

where

$$T \equiv 2/\omega_p^2 N(\Omega) > 0 \tag{3.25}$$

is the round-trip duration for the sheet launched at $t_o = \pi/\Omega$ (assuming no t_o cutoff -- see the next subsection). It follows that t_o^* given by

$$t_o^* = t_{rm}(t_o) \tag{3.26}$$

satisfies Eq. (3.11), with $t_o^* = \infty$ if $t_o = 0$, so Eqs. (3.19) - (3.21) are valid, given $t_o \in [0, \pi/\Omega]$, for

$$t \in [t_o, t_o + T/G(\Omega t_o)). \tag{3.27}$$

In summary, we may combine Eqs. (3.19) - (3.21) with Eq. (3.13) to arrive at expressions for position, velocity, and acceleration of the charge sheet with label $t_0 \in [0, \pi/\Omega]$, for all $t \in [0, \infty)$, as follows:

$$Z(t_o; t) = \begin{cases} v_o(t - t_o)[1 - T^{-1}G(\Omega t_o)(t - t_o)], & \text{if } t \in I(t_o) \\ 0, & \text{if } t \in I^-(t_o)\ , \end{cases} \tag{3.28}$$

$$V(t_o; t) = \begin{cases} v_o[1 - 2T^{-1}G(\Omega t_o)(t - t_o)], & \text{if } t \in I(t_o) \\ 0, & \text{if } t \in I^-(t_o)\ , \end{cases} \tag{3.29}$$

and

$$A(t_o;t) = \begin{cases} -2(v_o/T)G(\Omega t_o), & \text{if } t \in I(t_o) \\ 0, & \text{if } t \in I^-(t_o)\ , \end{cases} \tag{3.30}$$

where we have used

$$I(t_o) \equiv [t_o,\ t_o + T/G(\Omega t_o)] = [t_o, t_o^*] \tag{3.31}$$

and

$$I^-(t_o) \equiv (-\infty, \infty) \setminus I(t_o). \tag{3.32}$$

We will also denote, for future use,

$$I^o(t_o) = (t_o, t_o + T/G(\Omega t_o)) = (t_o, t_o^*)\ . \tag{3.33}$$

Note that $Z(t_o;\cdot)$ is continuous at t_o and t_o^* but neither $V(t_o;\cdot)$ nor $A(t_o;\cdot)$ is at either; further, we have arbitrarily elected to define these latter two maps at t_o^* so that they are left-continuous there (we have already arranged that they be right-continuous at t_o). Also,

$$V(t_o;t) = (\partial Z/\partial t)(t_o;t) \text{ and } A(t_o;t) = (\partial^2 Z/\partial t^2)(t_o;t)\ , \qquad t \neq t_o, t_o^*. \tag{3.34}$$

It follows from above and the previous paragraph that

$$|V(t_o;t)| \leq v_o, \qquad t \in I(t_o) \tag{3.35}$$

so that the motion is always non-relativistic if it is initially so.

C. The Cutoff

The results expressed in Eqs. (3.28) - (3.30) are constrained by the NCSC requirement. Since this constraint was imposed on the already existing equation of motion, Eq. (3.9), which does not *a priori* exclude charge sheet crossing, then we must insure that our complete model, including NCSC, is self-consistent. That such self-consistency does not automatically obtain in general may be seen as follows. First define quantity $\bar{t}_o$ by

$$\bar{t}_o \equiv \bar{t}_o[T, G] \equiv \sup \{T \in (0, \pi/\Omega] \,|\, (dt_{rm}/dt_o)(\tau) \le 0 \text{ for all } \tau \in (0, T]\}, \tag{3.36}$$

where $t_{rm}(t_0)$ is given by Eq. (3.24); note that

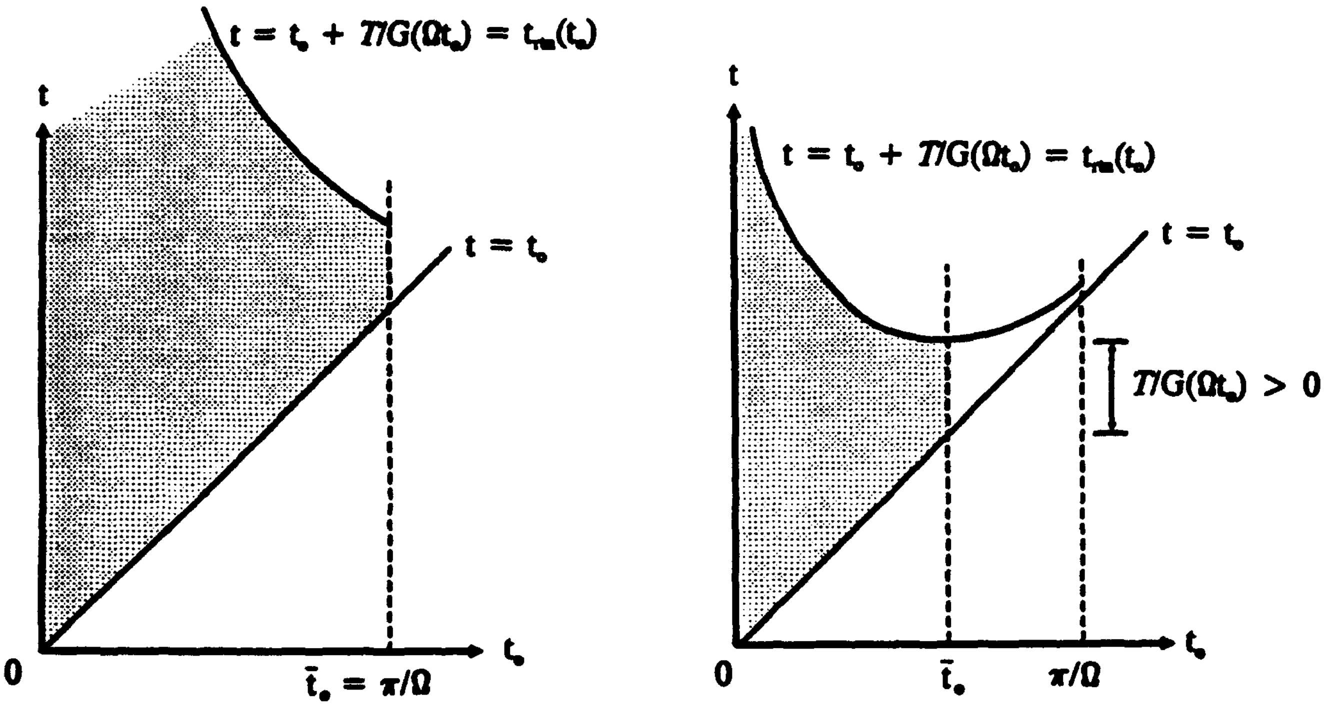
$$(dt_{rm}/dt_o)(t_o) = 1 - \Omega T[G'(\Omega t_o)/G^2(\Omega t_o)] \tag{3.37}$$

exists on $[0, \pi/\Omega]$ since G' does, that $\bar{t}_o \le \pi/\Omega$, and that $\bar{t}_o$ satisfies

$$G'(\Omega\bar{t}_o)/G^2(\Omega\bar{t}_o) = 1/\Omega T \tag{3.38}$$

whenever $\bar{t}_o < \pi/\Omega$. Now consider, in Fig. 2, two typical pictures in the t_0 - t plane of the set $\bigcup_{t_o \in [0,\pi/\Omega]} [\{t_o\} \times I(t_o)]$ of arguments for which sheet motion is, according to Eqs. (3.28) - (3.30), non-stationary. The behavior of t_{rm} near $t_o = 0$ follows from $\lim_{t_o \to 0^+} t_{rm}(t_o) = \infty$ which in turn follows from $\lim_{t_o \to 0^+} G(\Omega t_o) = \infty$, while the behavior elsewhere follows from the fact that G is strictly increasing. Then it is clear that NCSC does not hold in situations represented by Fig. 2b.

To rectify this situation, we have two options: We may either restrict ourselves to considering only pulses for which the behavior in Fig. 2a is representative, which behavior is characterized by LHS (3.37) ≤ 0 for $t_o \in [0, \pi/\Omega]$; or we may consider, in addition, pulses represented by Fig. 2b if we *cut off* from consideration in the dynamics (and hence subsequently in the electrodynamics) those



(a) $\bar{t}_o = \pi/\Omega$ (b) $\bar{t}_o < \pi/\Omega$

Figure 2. Typical domains in the t_o–t plane for non-stationary sheet behavior as determined by Eqs. (3.28)-(3.30).

charge sheets with $t_0 > \bar{t}_o$. (In fact, we could cut off somewhat beyond t_0 [2] but choose not to do so since we are interested in all $t > 0$.) Elimination of these late-time charge sheets does not at all influence the dynamics of the retained earlier-time sheets with labels $t_0 \le \bar{t}_o$ since the dynamics of any particular charge sheet is influenced only by those charge sheets emitted earlier than it (in the NCSC approximation). Further, when $\bar{t}_o$ does not differ greatly from π/Ω then using this cutoff is not an unreasonable approximation. In the literature, this issue is addressed only in [7] where the first option is chosen. We choose the second since it is more general than the first, including the latter as a special case. Naturally, we call $\bar{t}_o$ the *cutoff* on t_0. Finally, since we have based our strategy on Fig. 2, we should point out that the two possibilities illustrated there are not only typical but in fact general except for the possible occurence of points of inflection or intervals of constancy in $t_m(t_0)$, which features do not affect the above argument. Because such features will, however, have an influence on our radiation considerations, we demonstrate in Appendix A the existence of a class of pulses which have the interval-of-constancy feature.

While it is clear that elimination of charge sheets with $t_0 > \bar{t}_o$ is a necessary condition for self-consistency (when we wish to consider all $t > 0$), it is not *a priori* clear that it is also sufficient. That is, we may have to further limit those t_0's which we admit to the charge cloud in order to insure consistency with the NCSC constraint. We now settle this issue, showing that this condition is in fact also sufficient. To that end, we let label t_0 be *admissible* if charge sheet t_0, during its entire flight, never spatially coincides with any other earlier-launched charge sheet; i.e.:

Definition 3.1: $t_o \in [0, \pi/\Omega]$ is *admissible* if for every $t_o' \in [0, t_o)$ and for every $t \in \Gamma(t_o) \cap \Gamma(t_o')$ we have $Z(t_o'; t) \neq Z(t_o; t)$.

It is clear from this definition that $t_o = 0$ is always admissible (vacuously); physically this may also be seen since $Z(0; t) = v_o t$ for any pulse. In fact, as we demonstrate in Appendix B, the following is true:

Theorem 3.2: (i) $\bar{t}_o > 0$

(ii) If $t_o \in [0, \bar{t}_o]$ then t_o is admissible; further, $[0, \bar{t}_o]$ is the largest possible interval of admissible t_o's which contains $t_o = 0$.

In summary, the NCSC requirement has led us to limit (when $\bar{t}_o < \pi/\Omega$) those charge sheets, labeled by t_o, which we admit to the charge cloud. As a consequence, if we define

$$D[T, G] \equiv \bigcup_{t_o \in [0, \bar{t}_o]} [\{t_o\} \times I(t_o)] \tag{3.39}$$

as well as denote its complement and interior, respectively, by

$$D^{-}[T, G] = \{[0, \bar{t}_o] \times (-\infty, \infty)\} \setminus D[T, G] \tag{3.40}$$

and

$$D^{o}[T, G] = \bigcup_{t_o \in (0, \bar{t}_o)} [\{t_o\} \times I^{o}(t_o)] \ , \tag{3.41}$$

then the maps $(t_o, t) \mapsto Z(t_o; t), V(t_o; t), A(t_o; t)$ of Eqs. (3.28) - (3.30) will henceforth be considered to have only domain $[0, \bar{t}_o] \times (-\infty, \infty)$ with their values on $D[T, G]$ being given by the respective first lines of Eqs. (3.28)-(3.30) and their values on $D^{-}[T, G]$ being 0. In particular, they will no longer be considered to be defined for values of t_o in $(\bar{t}_o, \pi/\Omega]$. We will present, in Section VII examples of pulses with $\bar{t}_o = \pi/\Omega$ and of pulses with $\bar{t}_o < \pi/\Omega$.

D. Charge and Current Densities

Since, as pointed out in the Introduction, our formulation of electrodynamics is driven by the inability to translate analytically from the Eulerian to the Lagrangian description of the electron cloud

charge density, $\rho(z, t)$, and current density,

$$J(z, t) = \rho(z, t)v(z, t) \tag{3.42}$$

(where $v(z, t)$ represents the charge cloud velocity at (z, t)), we devote some discussion to this issue.

Consider first the charge density. Given $z > 0$ and $t > 0$, we expect that

$$\rho(z, t) = (\partial\tau_o/\partial z)(z, t)\rho_L(\tau_o(z, t)) \tag{3.43}$$

where $t_o = \tau_o(z, t)$ is "the" label that satisfies

$$Z(t_o; t) = z \tag{3.44}$$

for $\rho_L(t_o)$ given by Eq. (3.5) and $Z(t_o; t)$ given by Eq. (3.28). This expectation is, of course, basically correct but we must be careful to ensure that the prescription given by Eq. (3.44) is well-defined, i.e., that such a t_o, given by a differentiable (wrt z) $\tau_o(z, t)$, exists and is unique. Indeed, Eq. (3.44) has, in general, many solutions for t_o (given z and t) and no valid physical solution for t_o whenever $z > v_o t$ (since such solutions would be acausal).

It is easy to settle the existence and uniqueness issue for Eq. (3.44). To this end, let $D_1[T, G]$ be the set defined in the z-t plane -- see Fig. 3 -- by

$$D_1[T, G] \equiv \{(\zeta, \tau) \mid \text{there exists } t_o \in [0, \bar{t}_o] \text{ such that } Z(t_o; \tau) = \zeta\}. \tag{3.45}$$

Clearly $D_1[T, G] \neq \emptyset$. Then the following, whose proof is provided in Appendix B, is true:

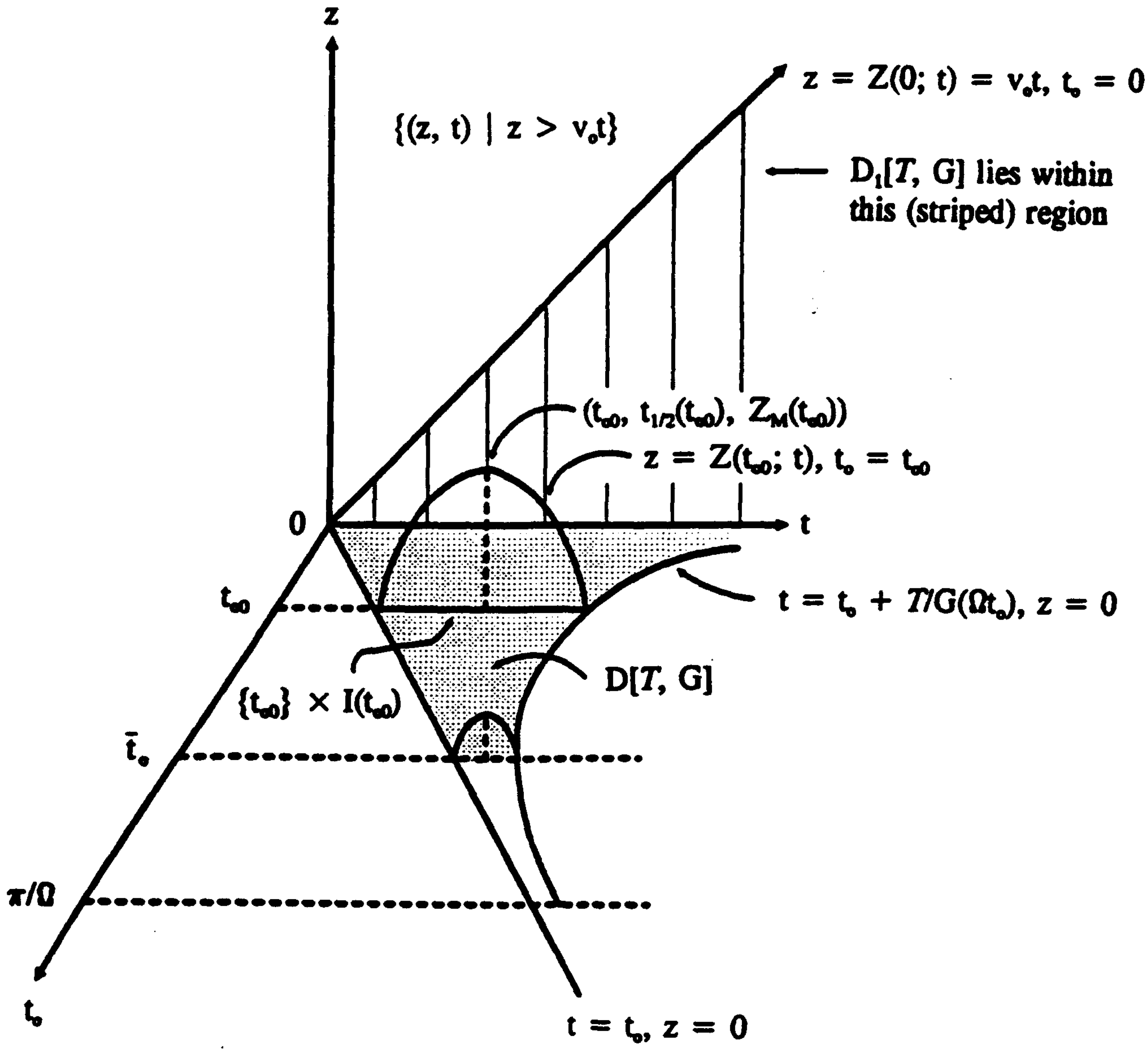


Figure 3. The extension of Fig. 2b to three dimensions to show some representative points on the surface $Z(t_o; t)$, $(t_o; t) \in D[T, G]$; t_{o0} is a typical value of t_o.

Theorem 3.3: Let z, t > 0. If $(z, t) \in D_1[T, G]$ then the equation $Z(t_o; t) = z$ has a unique solution in the interval $[0, \bar{T}_o]$; and if $(z, t) \notin D_1[T, G]$ then $\rho(z, t) = 0$. Further, if $z > 0$ and $t \leq 0$ then $\rho(z, t) = 0$.

The existence of $\partial\tau_o/\partial z$ at points $(z, t) \in D_1[T, G]$ follows from the Implicit Function Theorem; but, since we will not base our electrodynamics on $\rho(z,t)$ and $J(z,t)$, we will not give the details here. (However, the essentials of the argument are contained in a portion of the proof of Theorem C.4.)

Having found the appropriate $t_o = \tau_o(z, t)$ required for charge density in Eq. (3.43), we then find

$$v(z,t) = kV(\tau_o(z, t); t), \tag{3.46}$$

where $V(t_o; t)$ is given by Eq. (3.29), and finally find $J(z, t)$ from Eq. (3.42) so that

$$J(z, t) = k(\partial\tau_o/\partial z)(z, t)\rho_L(\tau_o(z, t))V(\tau_o(z, t); t). \tag{3.47}$$

In practice one finds t_o by solving Eq. (3.44) directly, either analytically or -- necessarily, in many cases -- numerically and by then judiciously selecting the correct t_o from among the in general many roots (all but one of which, however, lie outside $[0, \bar{T}_o]$). However, ρ and J are not generally explicitly analytically available *via* Eqs. (3.43) and (3.47) because Eq. (3.44) for t_o is not generally solvable analytically. This state-of-affairs, then, drives our choice of technique for doing, in the next section, charge cloud radiation electrodynamics.

IV. FINITELY-REMOTE FIELDS

A. Background

The usual methods for studying radiation theory for macroscopic charge and current distributions employ a field-theoretic (Eulerian) description of the distributions. In this approach, one assumes that $\rho(\mathbf{x}, t)$ and $\mathbf{J}(\mathbf{x}, t)$ are available and from them calculates the retarded potentials at space-time points $(\mathbf{x}, t)$ of interest according to

$$\phi_E(\mathbf{x}, t) = (1/4\pi\varepsilon_o)\iiint dV(\mathbf{x}')\rho(\mathbf{x}', t - (|\mathbf{x} - \mathbf{x}'|/c))|\mathbf{x} - \mathbf{x}'|^{-1} \tag{4.1}$$

and

$$\mathbf{A}_E(\mathbf{x}, t) = (1/4\pi\varepsilon_o c^2)\iiint dV(\mathbf{x}')\mathbf{J}(\mathbf{x}', t - (|\mathbf{x} - \mathbf{x}'|/c))|\mathbf{x} - \mathbf{x}'|^{-1} \tag{4.2}$$

where the integrals extend over any volume containing all the sources ($\mathbf{x}$ and $\mathbf{x}'$ are position vectors with respect to some arbitrary origin); the fields $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ then follow from

$$\mathbf{E}(\mathbf{x}, t) = -\nabla\phi(\mathbf{x}, t) - (\partial\mathbf{A}/\partial t)(\mathbf{x}, t) \quad \text{and} \quad \mathbf{B}(\mathbf{x}, t) = \nabla \times \mathbf{A}(\mathbf{x}, t). \tag{4.3}$$

All electrodynamic quantities of interest follow from $\mathbf{E}$ and $\mathbf{B}$; in particular, radiation quantities follow from those parts of $\mathbf{E}$ and $\mathbf{B}$ falling off with distance as $1/|\mathbf{x}|$ as opposed to $1/|\mathbf{x}|^2$.

As pointed out in detail at the end of the previous section, the above Eulerian description is not adequate and we must resort to another approach. This other approach is to use a particle-theoretic (Lagrangian) description of the macroscopic charge and current distributions which give rise to the radiation fields. This method, while common for studying the fields of a single moving charge, is not generally used for analytical treatments of macroscopic, continuously distributed aggregates of charged particles; but it is the method we use here. This Lagrangian point-of-view will allow us to

obtain analytical expressions for all the usual radiation quantities of interest without sacrificing the generality we require. In this approach, a macroscopic charge distribution is considered to be a union of "infinitesimal" charge elements, $\delta q(\ell)$, with label ℓ in some index set, $\mathfrak{L}$, where $\delta q(\ell)$ has position, velocity, and acceleration denoted by $\mathbf{X}(\ell; t)$, $\dot{\mathbf{X}}(\ell; t)$, and $\ddot{\mathbf{X}}(\ell; t)$ respectively. (The $\dot{}$ and $\ddot{}$ should be regarded here only as distinguishing notation -- we do not mean to imply that $t \mapsto \mathbf{X}(\ell; t)$ is twice differentiable on all of $(-\infty, \infty)$.) The potentials corresponding to Eqs. (4.1) and (4.2) are then given by

$$\phi_L(\mathbf{x}, t) = (1/4\pi\varepsilon_o)\int_{\mathfrak{L}} dq(\ell)\,|\mathbf{x} - \mathbf{X}(\ell; t'(\ell; \mathbf{x}, t))|^{-1}[1 - \Theta(\ell; \mathbf{x}, t)]^{-1} \tag{4.4}$$

and

$$\mathbf{A}_L(\mathbf{x}, t) = (1/4\pi\varepsilon_o c^2)\int_{\mathfrak{L}} dq(\ell)\dot{\mathbf{X}}(\ell; t'(\ell; \mathbf{x}, t))\,|\mathbf{x} - \mathbf{X}(\ell; t'(\ell; \mathbf{x}, t))|^{-1}[1 - \Theta(\ell; \mathbf{x}, t)]^{-1} \tag{4.5}$$

where

$$\Theta(\ell; \mathbf{x}, t) = (1/c)\dot{\mathbf{X}}(\ell; t'(\ell; \mathbf{x}, t)) \cdot \hat{\mathbf{R}}[\mathbf{x}, \mathbf{X}(\ell; t'(\ell; \mathbf{x}, t))]\ , \tag{4.6}$$

$$\hat{\mathbf{R}}[\mathbf{x}, \mathbf{X}(\ell; t'(\ell; \mathbf{x}, t))] = [\mathbf{x} - \mathbf{X}(\ell; t'(\ell; \mathbf{x}, t))]\ /\ |\mathbf{x} - \mathbf{X}(\ell; t'(\ell; \mathbf{x}, t))| \tag{4.7}$$

is the unit vector pointing from $\mathbf{X}(\ell; t'(\ell; \mathbf{x}; t))$ to $\mathbf{x}$, and $t'(\ell; \mathbf{x}, t)$ is the retarded time which is the solution of

$$t - t' = (1/c)\,|\mathbf{x} - \mathbf{X}(\ell; t')|, \quad t' \le t, \tag{4.8}$$

given ℓ, $\mathbf{x}$ and t. The fields $\mathbf{E}$ and $\mathbf{B}$ are then once again determined by Eq. (4.3). (It is assumed that

Eq. (4.8) has a unique, well-behaved solution for t', given ℓ, $\mathbf{x}$, t; we will have more to say about this later.) In case the distribution is a single point charge q (so that $\mathfrak{L} = \{1\}$ and the label ℓ may be suppressed, and $|\delta q(1)| = q$) the potentials ϕ_L and $\mathbf{A}_L$ are the Liénard-Wiechert potentials of the point charge.

Upon performing the operations indicated in Eq. (4.3) for **E** and **B** we find

$$\mathbf{E}(\mathbf{x},t) = (1/4\pi\varepsilon_o)\int_{\mathfrak{L}} dq(\ell)\ [\mathbf{\mathit{E}}_v(\ell;\mathbf{x},t'(\ell;\mathbf{x},t)) + \mathbf{\mathit{E}}_a(\ell;\mathbf{x},t'(\ell;\mathbf{x},t))] \tag{4.9}$$

and

$$\mathbf{B}(\mathbf{x},t) = (1/4\pi\varepsilon_o c)\int_{\mathfrak{L}} dq(\ell)\{\hat{\mathbf{R}}[\mathbf{x},\mathbf{X}(\ell;t'(\ell;\mathbf{x},t))] \times [\mathbf{\mathit{E}}_v(\ell;\mathbf{x},t'(\ell;\mathbf{x},t)) + \mathbf{\mathit{E}}_a(\ell;\mathbf{x},t'(\ell;\mathbf{x},t))]\} \tag{4.10}$$

where

$$\mathbf{\mathit{E}}_v(\ell;\mathbf{x},t'(\ell;\mathbf{x},t)) = [\hat{\mathbf{R}} - (1/c)\dot{\mathbf{X}}(\ell;t')]\ /\ \gamma^2(\ell,t')[1-\Theta(\ell;\mathbf{x},t)]^3|\mathbf{x}-\mathbf{X}(\ell;t')|^2, \tag{4.11}$$

$$c^2\mathbf{\mathit{E}}_a(\ell;\mathbf{x},t'(\ell;\mathbf{x},t)) = \left(\hat{\mathbf{R}} \times \{[\hat{\mathbf{R}} - (1/c)\dot{\mathbf{X}}(\ell;t')] \times \ddot{\mathbf{X}}(\ell;t')\}\right)\ /\ [1-\Theta(\ell;\mathbf{x},t)]^3|\mathbf{x}-\mathbf{X}(\ell;t')|, \tag{4.12}$$

and

$$\gamma(\ell,t') = \{1 - [|\dot{\mathbf{X}}(\ell;t')|/c]^2\}^{-1/2}; \tag{4.13}$$

in the above we have abbreviated by $\hat{\mathbf{R}}$ the quantity which is represented more fully in Eq. (4.7), and by t' the retarded time $t'(\ell;\mathbf{x},t)$. The subscripts "v" and "a" on $\mathbf{\mathit{E}}$ stand for "velocity" and "acceleration"; the velocity fields fall off as $1/|\mathbf{x}-\mathbf{X}|^2$ and are essentially static in character, while

the acceleration fields fall off as $1/|\mathbf{x} - \mathbf{X}|$ and are the dynamic radiation fields. In what follows we will be interested only in the radiation fields

$$\mathbf{E}_a(\mathbf{x}, t) = (1/4\pi\varepsilon_o) \int_{\mathbf{\Omega}} dq(\ell) \mathit{E}_a(\ell; \mathbf{x}, t'(\ell; \mathbf{x}, t)) \tag{4.14}$$

and

$$\mathbf{B}_a(\mathbf{x}, t) = (1/4\pi\varepsilon_o c) \int_{\mathbf{\Omega}} dq(\ell) \{ \hat{\mathbf{R}}[\mathbf{x}, \mathbf{X}(\ell; t'(\ell; \mathbf{x}, t))] \times \mathit{E}_a(\ell; \mathbf{x}, t'(\ell; \mathbf{x}, t)) \}. \tag{4.15}$$

The Poynting vector associated with this electromagnetic field is

$$\mathbf{S}_a(\mathbf{x}, t) = (\varepsilon_o c^2) \mathbf{E}_a(\mathbf{x}, t) \times \mathbf{B}_a(\mathbf{x}, t) \qquad \text{(watts/m}^2\text{)} \tag{4.16}$$

so that the instantaneous radiation energy flux area density at $\mathbf{x}$ in the direction $\hat{\mathbf{x}}$ is $\mathbf{S}_a(\mathbf{x}, t) \cdot \hat{\mathbf{x}}$ and the instantaneous radiation energy flux angular density at $\mathbf{x}$, $|\mathbf{x}| \gg 0$, in the direction $\hat{\mathbf{x}}$ (also called instantaneous radiated power per unit solid angle) is

$$(\delta P/\delta\Psi)(\mathbf{x}, t) = \mathbf{S}_a(\mathbf{x}, t) \cdot |\mathbf{x}|^2 \hat{\mathbf{x}} = \varepsilon_o c^2 \hat{\mathbf{x}} \cdot |\mathbf{x}| \mathbf{E}_a(\mathbf{x}, t) \times |\mathbf{x}| \mathbf{B}_a(\mathbf{x}, t) \qquad \text{(watts/sr)} \tag{4.17}$$

(where solid angle is denoted by Ψ). The total radiated energy angular density at $\mathbf{x}$ in the direction $\hat{\mathbf{x}}$ is

$$(\delta W/\delta\Psi)(\mathbf{x}) = \int_0^{\infty} (\delta P/\delta\Psi)(\mathbf{x}, t) dt = \int_{-\infty}^{\infty} (\delta P/\delta\Psi)(\mathbf{x}, t) dt \qquad \text{(joules/sr)} \tag{4.18}$$

and the spectral intensity of this radiation is given, with $\omega \geq 0$, by

$$(\delta^2 I/\delta\Psi\,\delta\omega)(\mathbf{x},\omega) = 2\varepsilon_o c^2|\mathbf{x}|^2 Re[\hat{\mathbf{x}}\cdot\mathbf{E}_{sA}(\mathbf{x},\omega)\times\mathbf{B}^*_{sA}(\mathbf{x},\omega)] \quad \text{(joules/sr-hz)} \tag{4.19}$$

where

$$\begin{Bmatrix}\mathbf{E}_{sA}(\mathbf{x},\omega)\\ \mathbf{B}_{sA}(\mathbf{x},\omega)\end{Bmatrix} = (2\pi)^{-1/2}\int_{-\infty}^{\infty}dt\; e^{i\omega t}\begin{Bmatrix}\mathbf{E}_s(\mathbf{x},t)\\ \mathbf{B}_s(\mathbf{x},t)\end{Bmatrix} = (2\pi)^{-1/2}\int_{0}^{\infty}dt\; e^{i\omega t}\begin{Bmatrix}\mathbf{E}_s(\mathbf{x},t)\\ \mathbf{B}_s(\mathbf{x},t)\end{Bmatrix} \tag{4.20}$$

and * denotes complex conjugate; Eq. (4.19) follows from the requirement

$$(\delta W/\delta\Psi)(\mathbf{x}) = \int_0^{\infty} d\omega(\delta^2 I/\delta\Psi\delta\omega)(\mathbf{x},\omega). \tag{4.21}$$

B. The Field integrals

Up to this point, everything we have done in this section has been very general. We now return to the particulars of our problem which are illustrated in Fig. 4. In that figure, each infinitesimal charge element of the electron cloud is labeled by the label t_o of the sheet on which it (forever) lies and by its plane polar coordinates, (r, ϑ), on that sheet which also remain constant in time. (r remains constant because of our assumption of negligible radial expansion while ϑ remains constant because of the cylindrical symmetry of the charge cloud.) Thus, our set of labels for the charge elements is

$$\mathfrak{L} = \{(t_o, r, \vartheta)\,|\,t_o \in [0,\bar{t}_o],\ r \in [0,a],\ \vartheta \in [-\pi,\pi) \text{ if } r \neq 0, \text{ and } \vartheta = 0 \text{ if } r = 0\}. \tag{4.22}$$

Using Eqs. (3.1) and (3.18), we have

$$\begin{aligned}\delta q(t_o, r, \vartheta) &= [\delta q(t_o)/\pi a^2]\,r dr d\vartheta = -eYA\Omega N(\Omega)G'(\Omega t_o)\delta t_o r dr d\vartheta \\ &= -(m_e\varepsilon_o/e)v_o\omega_p^2\Omega N(\Omega)G'(\Omega t_o)\,\delta t_o r dr d\vartheta\ ;\end{aligned} \tag{4.23}$$

also

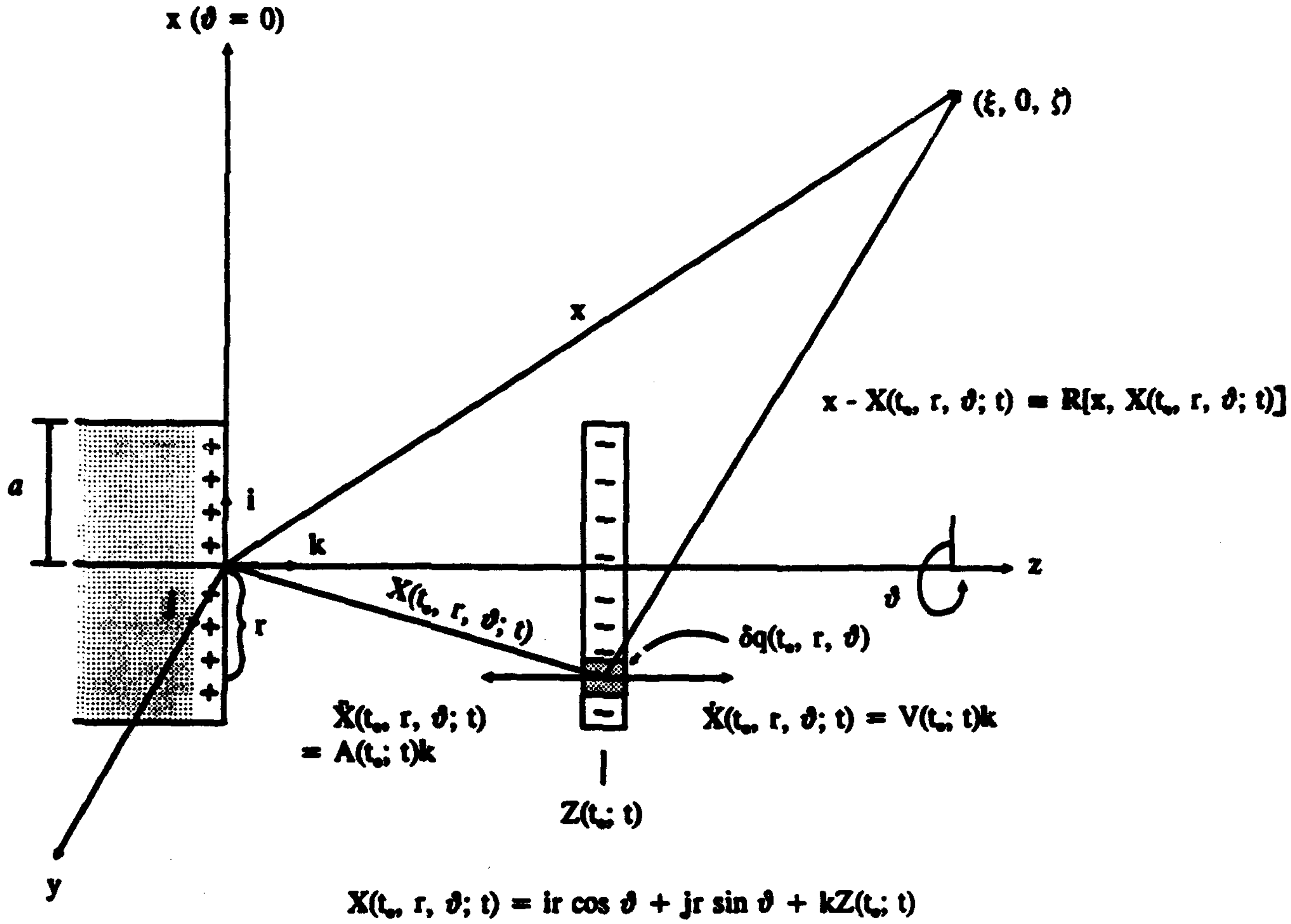


Figure 4. Electron cloud description for electrodynamics. The symbol t may be interpreted as present or retarded time, as needed. The vector **x** is taken to lie in the first quadrant of the x-z plane and outside the cloud (hence $\mathbf{x} \cdot \mathbf{i} > a$).

$$\mathbf{X}(t_o, r, \vartheta; t) = \mathbf{i}\ r \cos\vartheta + \mathbf{j}\ r \sin\vartheta + \mathbf{k}\ Z(t_o; t), \tag{4.24}$$

$$\dot{\mathbf{X}}(t_o, r, \vartheta; t) = \mathbf{k}\ V(t_o; t) \tag{4.25}$$

and

$$\ddot{\mathbf{X}}(t_o, r, \vartheta; t) = \mathbf{k}\ A(t_o; t) \tag{4.26}$$

where Z, V and A are given by Eqs. (3.28)-(3.30) with $t \in (-\infty, \infty)$.

We now write our expressions for the fields $\mathbf{E}_a(\mathbf{x}, t)$ and $\mathbf{B}_a(\mathbf{x}, t)$ based on Eqs. (4.14) and (4.15). Since we have assumed that v_o is non-relativistic, it follows from Eq. (3.35) that so is $V(t_o; t)$ for all $(t_o, t) \in [0, \bar{t}_o] \times (-\infty, \infty)$; hence we take in Eq. (4.12)

$$[1 - \Theta(t_o, r, \vartheta; \mathbf{x}, t)]^{-3} = 1 + 3(1/c)\dot{\mathbf{X}}(t_o, r, \vartheta; t'(t_o, r, \vartheta; \mathbf{x}, t)) \cdot \hat{\mathbf{R}}[\mathbf{x}, \mathbf{X}(t_o, r, \vartheta; t'(t_o, r, \vartheta; \mathbf{x}, t))]. \tag{4.27}$$

Denoting

$$\chi_D(t_o; t) = \begin{cases} 1, & \text{if } (t_o, t) \in D[T, G] \\ 0, & \text{if } (t_o, t) \in D^-[T, G] \end{cases} \tag{4.28}$$

and noting from Eqs. (4.25) and (4.26) that $\dot{\mathbf{X}}(t_o, r, \vartheta; t) \times \ddot{\mathbf{X}}(t_o, r, \vartheta; t) = 0$ we find, for $t \geq 0$,

$$\mathbf{E}_a(\mathbf{x}, t) = K \int_0^{\bar{t}_o} dt_o G'(\Omega t_o) G(\Omega t_o) \int_0^a dr\, r \int_{-\pi}^{\pi} d\vartheta\, \chi_D(t_o; t')\{[(\hat{\mathbf{R}} \cdot \mathbf{k})\hat{\mathbf{R}} - \mathbf{k}][1 + W(t_o; t')] / |\mathbf{x} - \mathbf{X}(t_o, r, \vartheta; t')|\} \tag{4.29}$$

where $t' = t'(t_o, r, \vartheta; x, t)$ is retarded time (possibly $t' < 0$) satisfying

$$t - t' = (1/c)|x - X(t_o, r, \vartheta; t')| , \quad t' \leq t; \tag{4.30}$$

also

$$\hat{R} = \hat{R}[x, X(t_o, r, \vartheta; t')], \tag{4.31}$$

$$W(t_o; t') = 3(1/c)V(t_o; t')(\hat{R} \cdot k) \tag{4.32}$$

and

$$K = (m_e/\pi e c^2)\Omega(v_o/T)^2 . \tag{4.33}$$

Likewise, for $t \geq 0$,

$$B_a(x, t) = -(K/c) \int_0^{T_o} dt_o G'(\Omega t_o) G(\Omega t_o) \int_0^a dr\, r \int_{-\pi}^{\pi} d\vartheta \chi_D(t_o; t')\{(\hat{R} \times k)[1 + W(t_o; t')] / |x - X(t_o, r, \vartheta; t')|\} . \tag{4.34}$$

Note that the factor $\chi_D(t_o; t')$ ensures that sheet t_o contributes to the fields only if it is in flight at the retarded time t', i.e., if $t' \in I(t_c)$. It is clear at this point that

$$E_a(x, t) = 0 = B_a(x, t) \text{ for } 0 \leq t \leq |x|/c \tag{4.35}$$

although we will re-derive this result in a more systematic way later.

To proceed further, we concentrate upon the integrands in the above expressions. We remark

that by symmetry we may, without loss of generality, take **x** to lie in the first quadrant of the x-z plane so that

$$\mathbf{x}\cdot\mathbf{j} = 0. \tag{4.36}$$

Also, the observation point **x** is assumed to lie outside the cloud; hence

$$\xi \equiv \mathbf{x}\cdot\mathbf{i} > a > 0. \tag{4.37}$$

With a view towards addressing the components of $\mathbf{E}_a$ and $\mathbf{B}_a$ we compute, using

$$\zeta \equiv \mathbf{x}\cdot\mathbf{k}, \tag{4.38}$$

the components of the integrands to be (abbreviating $\mathbf{X} = \mathbf{X}(t_o, r, \vartheta; t')$)

$$\begin{aligned}[(\hat{\mathbf{R}}\cdot\mathbf{k})\hat{\mathbf{R}}\cdot\mathbf{i}] - (\mathbf{k}\cdot\mathbf{i}) &= (1/|\mathbf{x}-\mathbf{X}|^2)\,[(\mathbf{x}-\mathbf{X})\cdot\mathbf{k}]\,[(\mathbf{x}-\mathbf{X})\cdot\mathbf{i}] \\ &= (1/|\mathbf{x}-\mathbf{X}|^2)\,[\xi\zeta-\zeta(\mathbf{X}\cdot\mathbf{i}) - \xi(\mathbf{X}\cdot\mathbf{k}) + (\mathbf{X}\cdot\mathbf{i})(\mathbf{X}\cdot\mathbf{k})] \\ &= (1/|\mathbf{x}-\mathbf{X}|^2)\,[\xi\zeta-\zeta r\cos\vartheta - \xi Z(t_o;t) + Z(t_o;t) r\cos\vartheta] \\ &= (1/|\mathbf{x}-\mathbf{X}|^2)\,[Z(t_o;t) - \zeta](r\cos\vartheta - \xi),\end{aligned} \tag{4.39}$$

$$[(\hat{\mathbf{R}}\cdot\mathbf{k})\hat{\mathbf{R}}\cdot\mathbf{j}] - (\mathbf{k}\cdot\mathbf{j}) = (1/|\mathbf{x}-\mathbf{X}|^2)[Z(t_o;t) - \zeta] r\sin\vartheta, \tag{4.40}$$

$$[(\hat{\mathbf{R}}\cdot\mathbf{k})\hat{\mathbf{R}}\cdot\mathbf{k}] - (\mathbf{k}\cdot\mathbf{k}) = (1/|\mathbf{x}-\mathbf{X}|^2)[Z(t_o;t) - \zeta]^2 - 1\,, \tag{4.41}$$

$$(\hat{\mathbf{R}}\times\mathbf{k})\cdot\mathbf{i} = -(1/|\mathbf{x}-\mathbf{X}|)(\mathbf{X}\cdot\mathbf{j}) = -(1/|\mathbf{x}-\mathbf{X}|) r\sin\vartheta, \tag{4.42}$$

$$(\hat{\mathbf{R}} \times \mathbf{k})\cdot\mathbf{j} = (1/|\mathbf{x}-\mathbf{X}|)[(\mathbf{X}\cdot\mathbf{i}) - \xi] = (1/|\mathbf{x}-\mathbf{X}|)(r\cos\vartheta - \xi)\,, \tag{4.43}$$

and

$$(\hat{\mathbf{R}} \times \mathbf{k})\cdot\mathbf{k} = 0, \tag{4.44}$$

where

$$\begin{aligned}
|\mathbf{x}-\mathbf{X}|^2 &= |\mathbf{x}|^2 + |\mathbf{X}|^2 - 2\mathbf{x}\cdot\mathbf{X} = |\mathbf{x}|^2 + r^2 + Z^2(t_o;t) - 2[\xi r\cos\vartheta + \zeta Z(t_o;t)] \\
&= |\mathbf{x}|^2 - \zeta^2 + r^2 + [Z(t_o;t) - \zeta]^2 - 2\xi r\cos\vartheta \\
&= \xi^2 + r^2 - 2\xi r\cos\vartheta + [Z(t_o;t) - \zeta]^2.
\end{aligned} \tag{4.45}$$

Denoting

$$Z_*(t_o;t) \equiv Z(t_o;t) - \zeta \tag{4.46}$$

so that

$$W(t_o;t') = -3(1/c)V(t_o;t')Z_*(t_o;t')/|\mathbf{x}-\mathbf{X}| \tag{4.47}$$

we then have

$$\mathbf{E}_a(\mathbf{x},t)\cdot\mathbf{i} = K\int_0^{\bar{t}_o} dt_o G'(\Omega t_o)G(\Omega t_0)\int_0^a dr\ r\int_{-\pi}^{\pi} d\vartheta \chi_D(t_o;t')Z_*(t_o;t')[1+W(t_o;t')](r\cos\vartheta-\xi)/|\mathbf{x}-\mathbf{X}|^3, \tag{4.48}$$

$$\mathbf{E}_a(\mathbf{x},t)\cdot\mathbf{j} = K\int_0^{\bar{t}_o} dt_o G'(\Omega t_o)G(\Omega t_0)\int_0^a dr\ r\int_{-\pi}^{\pi} d\vartheta \chi_D(t_o;t')Z_*(t_o;t')[1+W(t_o;t')]r\sin\vartheta/|\mathbf{x}-\mathbf{X}|^3, \tag{4.49}$$

$$\mathbf{E}_a(\mathbf{x},t)\cdot\mathbf{k} = K\int_0^{\bar{t}_o} dt_o G'(\Omega t_o)G(\Omega t_0)\int_0^a dr\ r\int_{-\pi}^{\pi} d\vartheta \chi_D(t_o;t')\{[Z_*^2(t_o;t')/|\mathbf{x}-\mathbf{X}|^3]-[1/|\mathbf{x}-\mathbf{X}|]\}[1+W(t_o;t')], \tag{4.50}$$

$$\mathbf{B}_a(\mathbf{x},t)\cdot\mathbf{i} = (K/c)\int_0^{\bar{t}_o} dt_o G'(\Omega t_o)G(\Omega t_0)\int_0^a dr\ r\int_{-\pi}^{\pi} d\vartheta \chi_D(t_o;t')[1+W(t_o;t')]r\sin\vartheta/|\mathbf{x}-\mathbf{X}|^2, \tag{4.51}$$

$$\mathbf{B}_a(\mathbf{x},t)\cdot\mathbf{j} = -(K/c)\int_0^{\bar{t}_o} dt_o G'(\Omega t_o)G(\Omega t_0)\int_0^a dr\ r\int_{-\pi}^{\pi} d\vartheta \chi_D(t_o;t')[1+W(t_o;t')](r\cos\vartheta-\xi)/|\mathbf{x}-\mathbf{X}|^2, \tag{4.52}$$

and

$$\mathbf{B}_a(\mathbf{x},t)\cdot\mathbf{k} = 0. \tag{4.53}$$

We next want to perform the r-ϑ integrations, but we are hindered by the fact that t' is a function of r and ϑ. Indeed, we have from Eqs. (4.30), (4.24), and (3.28) that t' satisfies, for $t_o \in [0, \bar{t}_o]$,

$$t - t' = \begin{cases} (1/c)\{\xi^2+r^2-2\xi r\cos\vartheta+[v_o(t'-t_o)-(v_0/T)G(\Omega t_o)(t'-t_o)^2-\zeta]^2\}^{1/2}, & \text{if } t'\in I(t_o) \\ (1/c)\{\xi^2+r^2-2\xi r\cos\vartheta+\zeta^2\}^{1/2}, & \text{if } t'\in I^-(t_o) \end{cases} \tag{4.54}$$

(the non-negative square root is assumed). Defining

$$\Lambda_*(t_o;t') \equiv Z_*(t_o;t')/\xi, \tag{4.55}$$

we may rewrite Eq. (4.54) as

$$t - t' = \begin{cases} (\xi/c)(1+\Lambda_o^2)^{1/2}\{1-[2/(1+\Lambda_o^2)](r/\xi)\cos\vartheta + [1/(1+\Lambda_o^2)](r/\xi)^2\}^{1/2}, & \text{if } t' \in I(t_o) \\ (|\mathbf{x}|/c)\left[1-2(\xi/|\mathbf{x}|)^2(r/\xi)\cos\vartheta + (\xi/|\mathbf{x}|)^2(r/\xi)^2\right]^{1/2}, & \text{if } t' \in I^-(t_o)\ . \end{cases} \tag{4.56}$$

If we now restrict our attention only to situations for which

$$a/\xi < 1 \tag{4.57}$$

and further demand only zeroth order accuracy in (r/ξ) then Eq. (4.54) becomes

$$t - t' = \begin{cases} (\xi/c)[1 + \Lambda_o^2(t_o;t')]^{1/2}, & \text{if } t' \in I(t_o) \\ |\mathbf{x}|/c, & \text{if } t' \in I^-(t_o) \end{cases} \tag{4.58}$$

with solution $\bar{t}' = \bar{t}'(t_o;\mathbf{x},t)$ (that solutions to Eq. (4.58) exist and are unique for fixed $t_o \in [0,\bar{t}_o]$, $\mathbf{x} = (\xi > 0,\ 0,\ \zeta \geq 0)$, and $t \geq 0$ will be demonstrated in Appendix C) which is an approximation to the exact solution $t' = t'(t_o, r, \vartheta; \mathbf{x}, t)$ of Eq. (4.54) but which is independent of r and ϑ.

The condition of Eq. (4.57) guarantees that $|\delta t|(t_o, r, \vartheta; \mathbf{x}, t) \equiv |t'(t_o, r, \vartheta; \mathbf{x}, t) - \bar{t}'(t_o; \mathbf{x}, t)|$ is small, but it does not guarantee that this difference is small enough, where "small enough" means, for example,

$$\sup_{t_o \in [0,\bar{t}_o]} \ \sup_{r \in [0,a],\ \vartheta \in [-\pi,\pi)} [v_o|\delta t|(t_o, r, \vartheta; \mathbf{x}, t)/Z_M(t_o)] < 1, \tag{4.59}$$

this being a sufficient condition that every sheet moves only a small fraction of its entire trajectory during the time $|t' - \bar{t}'|$. When Eq. (4.59) holds, then $|Z_o(t_o;t') - Z_o(t_o;\bar{t}')|/Z_M(t_o)$ and $|V(t_o;t') - V(t_o;\bar{t}')|/v_o$ are very small and $|\chi_D(t_o;t') - \chi_D(t_o;\bar{t}')| = 1$ for only a very small fraction of $t_o \in [0, \bar{t}_o]$, hence the integrals in Eqs. (4.48)-(4.52) remain quite accurate when t' is

replaced with $\bar{t}'$. That some such additional condition is needed is clear since without it the following unreasonable argument would be valid: Since we are ultimately interested in $\xi \to \infty$ for the radiation limit then, given *any* $a > 0$, consider only ξ so large that $a/\xi \ll 1$; hence *any* $a > 0$ can be accommodated by the condition $a/\xi \ll 1$! The condition of Eq. (4.59) further restricts a so that the above unreasonable argument is in fact not valid. Indeed, noting that using $\bar{t}'$ is equivalent to assuming that all charge elements dS on sheet t_o have one and the same associated retarded time, namely that associated with the charge element at $r = 0$, we see that we may approximate $|\delta t|(t_o, r, \vartheta; \mathbf{x}, t) \leq a/c$; further, from Eq. (3.22) we see that $Z_M(t_o) \geq 1/4\ v_o T$ for all $t_o \in [0, \bar{t}_o]$. Hence a sufficient condition for Eq. (4.59) to hold is that $v_o(a/c)/(1/4)v_o T \ll 1$, i.e.,

$$a/cT \ll 1, \tag{4.60}$$

and we require in the sequel that a satisfy this condition as well as that of Eq. (4.57).

Taken together, Eqs. (4.57) and (4.60) simply mean that the charge cloud has negligible radial extent for retardation purposes. (In the language of optics, we are discarding phase differences, which lead to interference effects, along the cross-section of the spot.) For this reason we refer to $\bar{t}'$ as the ***small spot size retarded time***. Note that we have made no assumption about the magnitude of Λ, (which is related to the axial extent of the charge cloud).

We may now move χ_D, Z_*, and the numerator of W in each of Eqs. (4.48) - (4.52) outside the r-ϑ integrals, replacing t' by $\bar{t}'$. When we do so, and further note that the integrands remaining under the r-ϑ integrals in the expressions for $\mathbf{E}_a \cdot \mathbf{j}$ and $\mathbf{B}_a \cdot \mathbf{i}$ are odd functions of ϑ, we get

$$\mathbf{E}_a(\mathbf{x},t)\cdot\mathbf{j} = 0 = \mathbf{B}_a(\mathbf{x},t)\cdot\mathbf{i},$$

$$\begin{aligned} \mathbf{E}_a(\mathbf{x},t)\cdot\mathbf{i} = 2K \int_0^{\bar{t}_o} dt_o G'(\Omega t_o) G(\Omega t_o) \chi_D(t_o;\bar{t}') \{ Z_*(t_o;\bar{t}') \int_0^a dr\, r \int_0^\pi d\vartheta (r\cos\vartheta - \xi)/|\mathbf{x}-\mathbf{X}|^3 \\ - (3/c) V(t_o;\bar{t}') Z_*^2(t_o;\bar{t}') \int_0^a dr\, r \int_0^\pi d\vartheta (r\cos\vartheta - \xi)/|\mathbf{x}-\mathbf{X}|^4 \}, \end{aligned} \tag{4.61}$$

$$\mathbf{E}_a(\mathbf{x},t)\cdot\mathbf{k} = 2K\int_0^{\tau_0} dt_o G'(\Omega t_o)G(\Omega t_0)\chi_D(t_o;\bar{t}')\{Z_e^2(t_o;\bar{t}')\int_0^a dr\, r\int_0^\pi d\vartheta(1/|\mathbf{x}-\mathbf{X}|^3) - \int_0^a dr r\int_0^\pi d\vartheta(1/|\mathbf{x}-\mathbf{X}|)$$

$$- (3/c)V(t_o;\bar{t}')[Z_e^3(t_o;\bar{t}')\int_0^a dr\, r\int_0^\pi d\vartheta(1/|\mathbf{x}-\mathbf{X}|^4) - Z_e(t_o;\bar{t}')\int_0^a dr\, r\int_0^\pi d\vartheta(1/|\mathbf{x}-\mathbf{X}|^2)]\} ,$$

(4.62)

and

$$\mathbf{B}_a(\mathbf{x},t)\cdot\mathbf{j} = -(2K/c)\int_0^{\tau_0} dt_o G'(\Omega t_o)G(\Omega t_0)\chi_D(t_o;\bar{t}')\{ \int_0^a dr\, r\int_0^\pi d\vartheta(r\cos\vartheta-\xi)/|\mathbf{x}-\mathbf{X}|^2$$

$$- (3/c)V(t_o;\bar{t}')Z_e(t_o;\bar{t}')\int_0^a dr\, r\int_0^\pi d\vartheta(r\cos\vartheta-\xi)/|\mathbf{x}-\mathbf{X}|^3\} . \qquad (4.63)$$

We see that $\mathbf{E}_a(\mathbf{x},t)$ lies in the plane determined by $\mathbf{x}$ and $\mathbf{k}$, while $\mathbf{B}_a(\mathbf{x},t)$ is perpendicular to that plane; the polarization of the electromagnetic field is thus specified.

We must next address the remaining r-ϑ integrations. While some of these double integrals can be evaluated analytically in closed form, the integrated results seem to us to be too complex for our further use here (i.e., integration over t_o) and we will not display them. Rather, we once again invoke the requirement $a/\xi < 1$ of Eq. (4.57), this time to expand the denominators in Eqs. (4.61) - (4.63). We first write, as in the RHS of Eq. (4.56),

$$|\mathbf{x}-\mathbf{X}| = \xi(1+\Lambda_e^2)^{1/2}\{1 - [2/(1+\Lambda_e^2)](r/\xi)\cos\vartheta + [1/(1+\Lambda_e^2)](r/\xi)^2\}^{1/2} \qquad (4.64)$$

so that, for n = 1, 2, 3, 4, and $a/\xi < 1$,

$$|\mathbf{x}-\mathbf{X}|^{-n} = \xi^{-n}(1+\Lambda_e^2)^{-n/2}\Big(1 + [n/(1+\Lambda_e^2)]\cos\vartheta\,(r/\xi) - [n/2(1+\Lambda_e^2)]\{1-[(n+2)/(1+\Lambda_e^2)]\cos^2\vartheta\}(r/\xi)^2$$

$$+ O((r/\xi)^3)\Big).$$

(4.65)

Using these expressions in Eqs. (4.61) - (4.63) and performing a great deal of algebra, we find the double integral expressions in braces in those three equations integrate to be, respectively,

$$-(\pi/2)\xi(a/\xi)^2\{\Lambda_o(1+\Lambda_o^2)^{-3/2}[1-(\tfrac{3}{8})(4\Lambda_o^2-1)(1+\Lambda_o^2)^{-2}(a/\xi)^2]$$
$$-3(V/c)\Lambda_o^2(1+\Lambda_o^2)^{-2}[1-(\tfrac{3}{4})(2\Lambda_o^2-1)(1+\Lambda_o^2)^{-2}(a/\xi)^2]\}, \tag{4.66}$$

$$-(\pi/2)\xi(a/\xi)^2\{(1+\Lambda_o^2)^{-3/2}[1+(\tfrac{1}{8})(10\Lambda_o^4-4\Lambda_o^2+1)(1+\Lambda_o^2)^{-2}(a/\xi)^2]$$
$$-3(V/c)\Lambda_o(1+\Lambda_o^2)^{-2}[1+(\tfrac{1}{2})(3\Lambda_o^4-2\Lambda_o^2+1)(1+\Lambda_o^2)^{-2}(a/\xi)^2]\}, \tag{4.67}$$

and

$$-(\pi/2)\xi(a/\xi)^2\{(1+\Lambda_o^2)^{-1}[1-\Lambda_o^2(1+\Lambda_o^2)^{-2}(a/\xi)^2]$$
$$-3(V/c)\Lambda_o(1+\Lambda_o^2)^{-3/2}[1-(\tfrac{3}{8})(4\Lambda_o^2-1)(1+\Lambda_o^2)^{-2}(a/\xi)^2]\}, \tag{4.68}$$

correct to order $(a/\xi)^2$ inside the braces of these last three equations. In what follows we will take only the leading term inside each of the brackets (i.e., 1) which, as can be seen, gives results correct to order $(a/\xi)^3$; these third order results are more than sufficient for our purpose. The fields then become

$$\mathbf{E}_a(\mathbf{x},t) = -(\pi a^2K/\xi)\int_0^{\tau_o} dt_o G'(\Omega t_o)G(\Omega t_o)\chi_D(t_o;\bar{t}')$$
$$\times\ \{[1+\Lambda_a^2(t_o;\bar{t}')]^{-3/2} - 3[V(t_o;\bar{t}')/c]\Lambda_a(t_o;\bar{t}')[1+\Lambda_a^2(t_o;\bar{t}')]^{-2}\}[\mathbf{i}\Lambda_a(t_o;\bar{t}') + \mathbf{k}] \tag{4.69}$$

and

$$\mathbf{B}_a(\mathbf{x},t) = \mathbf{j}(\pi a^2K/c\xi)\int_0^{\tau_o} dt_o G'(\Omega t_o)G(\Omega t_o)\chi_D(t_o;\bar{t}')\{[1+\Lambda_a^2(t_o;\bar{t}')]^{-1}$$
$$-\ 3[V(t_o;\bar{t}')/c]\Lambda_a(t_o;\bar{t}')[1+\Lambda_a^2(t_o;\bar{t}')]^{-3/2}\} \tag{4.70}$$

where

$$\Lambda_a(t_o;\bar{t}') = \begin{cases} [v_o(\bar{t}'-t_o) - (v_o/T)G(\Omega t_o)(\bar{t}'-t_o)^2]/\xi - (\zeta/\xi), & \text{if } \bar{t}' \in I(t_o) \\ -(\zeta/\xi), & \text{if } \bar{t}' \in I^-(t_o) \end{cases} \tag{4.71}$$

and

$$\bar{t}' = \bar{t}'(t_o; \mathbf{x}, t). \tag{4.72}$$

To proceed further, we need some additional properties of the retarded time $\bar{t}'$. Because we will need, in addition to these properties, several other results concerning the retarded time $\bar{t}'$ in order to demonstrate the existence of the asymptotic radiation limit, we systematically treat $\bar{t}'$ in a mathematically careful manner in Appendix C. In order to get on with the current development, however, we simply note that as a consequence of Theorem C.3 we may rewrite the fields $\mathbf{E}_a$ and $\mathbf{B}_a$ of Eqs. (4.69) and (4.70) as

$$\begin{aligned}\mathbf{E}_a(\mathbf{x}, t) = -(K'/\xi) \int_0^{\overline{T}_o(\mathbf{x},t)} dt_o G'(\Omega t_o) G(\Omega t_o) \{[1 + \Lambda_e^2(t_o; \bar{t}'(t_o; \mathbf{x}, t))]^{-3/2} \\ - 3[V(t_o; \bar{t}'(t_o; \mathbf{x}, t))/c]\, \Lambda_e(t_o; \bar{t}'(t_o; \mathbf{x}, t))[1 + \Lambda_e^2(t_o; \bar{t}'(t_o; \mathbf{x}, t))]^{-2}\} [\mathbf{i}\Lambda_e(t_o; \bar{t}'(t_o; \mathbf{x}, t)) + \mathbf{k}]\end{aligned} \tag{4.73}$$

and

$$\begin{aligned}\mathbf{B}_a(\mathbf{x}, t) = \mathbf{j}(K'/c\xi) \int_0^{\overline{T}_o(\mathbf{x},t)} dt_o G'(\Omega t_o) G(\Omega t_o) \{[1 + \Lambda_e^2(t_o; \bar{t}'(t_o; \mathbf{x}, t))]^{-1} \\ - 3[V(t_o; \bar{t}'(t_o; \mathbf{x}, t))/c] \Lambda_e(t_o; \bar{t}'(t_o; \mathbf{x}, t))[1 + \Lambda_e^2(t_o; \bar{t}'(t_o; \mathbf{x}, t))]^{-3/2}\}\end{aligned} \tag{4.74}$$

where $\overline{T}_o(\mathbf{x}, t)$ is given by

$$\overline{T}_o(\mathbf{x}, t) = \begin{cases} 0, & \text{if } -\infty < t - |\mathbf{x}|/c \le 0 \\ t - |\mathbf{x}|/c, & \text{if } 0 < t - |\mathbf{x}|/c \le \bar{t}_o \\ \bar{t}_o, & \text{if } \bar{t}_o < t - |\mathbf{x}|/c < t_{rm}(\bar{t}_o) \\ t_1(t - |\mathbf{x}|/c), & \text{if } t_{rm}(\bar{t}_o) \le t - |\mathbf{x}|/c < \infty \end{cases} \tag{4.75}$$

with

$$t_1(t-|\mathbf{x}|/c) \equiv \inf t_{rm}^{-1}[t-|\mathbf{x}|/c], \tag{4.76}$$

for

$$t_{rm}^{-1}[t-|\mathbf{x}|/c] \equiv \{t_o \in (0,\bar{t}_0] \,|\, t_{rm}(t_o) = t-|\mathbf{x}|/c\}; \tag{4.77}$$

and

$$K' = \pi a^2 K = \pi(m_e/ec^2)(\pi/\Omega)^{-1}(av_o/T)^2 = \pi[\Omega N(\Omega)/2\pi]^2(e^2/\varepsilon_o m_e c)^2(m_e/e)(\pi/\Omega)a^2(AY)^2. \tag{4.78}$$

Figure 5 illustrates the determination of $\overline{T}_o(\mathbf{x}, t)$, for the various ranges of $t - |\mathbf{x}|/c$ on the RHS of Eq. (4.75), as implied by Theorem C.3, while Fig. 6 illustrates a generic $\overline{T}_o(\mathbf{x}, t)$ as a function of $t \in (-\infty, \infty)$ for a fixed $\mathbf{x}$. Appendix C contains the background required to more fully appreciate these figures. Note that if $t_1(t - |\mathbf{x}|/c) < t_s(t - |\mathbf{x}|/c)$ (in case $t - |\mathbf{x}|/c \geq t_{rm}(\bar{t}_o)$), where

$$t_s(t-|\mathbf{x}|/c) = \sup t_{rm}^{-1}[t-|\mathbf{x}|/c], \tag{4.79}$$

then we must add $[t_1, t_s]$ to the ranges of integration in Eqs. (4.73) and (4.74) (for elaboration of this last point, see both the paragraph following Eq. (C.30) and the beginning of the next section). Also note that, as expected (see Eq. (4.35)), we have from Eq. (4.75) that

$$\mathbf{E}_a(\mathbf{x}, t) = \mathbf{0} = \mathbf{B}_a(\mathbf{x}, t) \text{ if } t \leq |\mathbf{x}|/c. \tag{4.35}$$

The final impediment to the evaluation of these integrals appears to be this: we do not yet have an explicit expression for $\bar{t}'(t_o; \mathbf{x}, t)$ when $t_o > 0$ ($\bar{t}'(0; \mathbf{x}, t)$ is given explicitly by Eq. (C.3)).

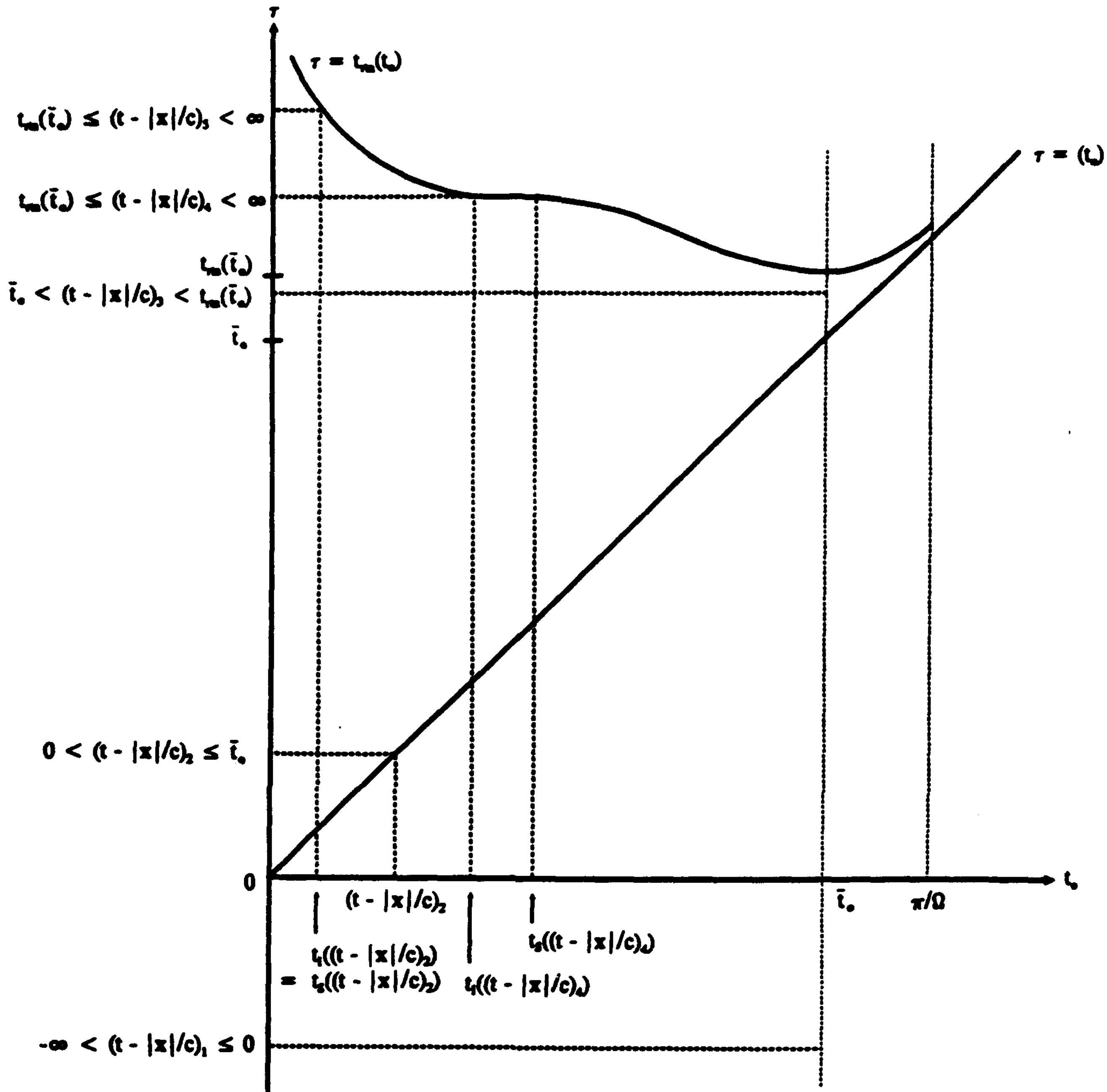


Figure 5. Various possibilities for $t - |x|/c$ (where $(t - |x|/c)_i$, $i = 1,\ldots,5$, indicates five specific choices for the value of variable $t - |x|/c$); and the ranges of values of t_o for which $(t_o, \bar{t}'(t_o; (x, t)_i)) \in D^o[T, G]$ according to Theorem C.3. The time variable is denoted here by τ to distinguish it from t.

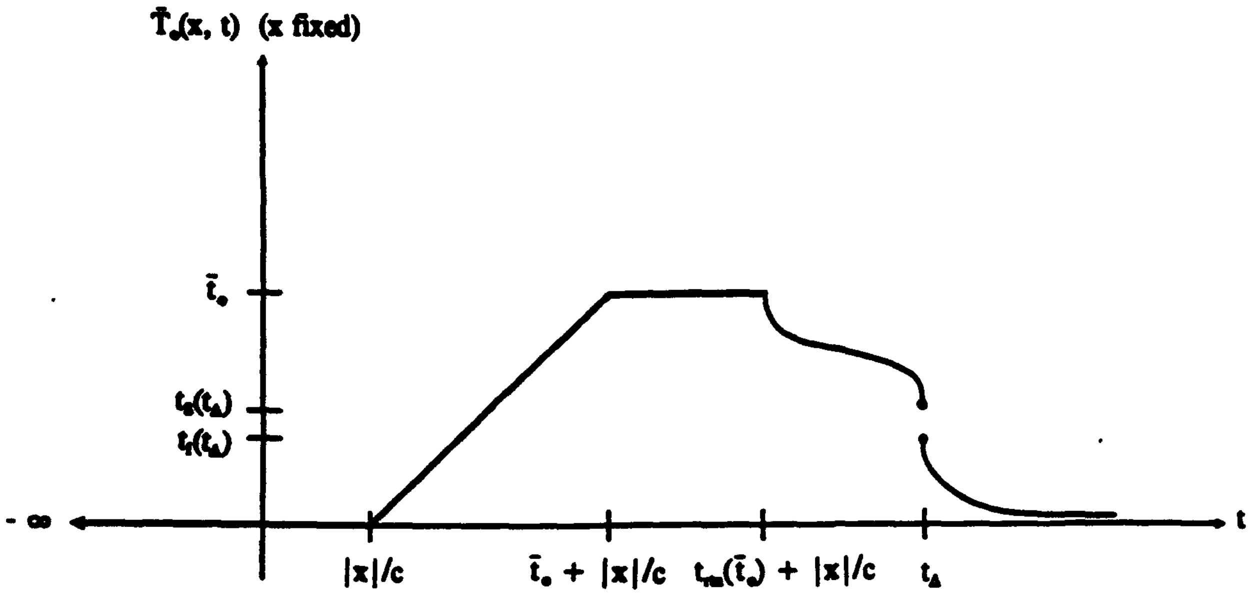


Figure 6. The function $\overline{T}_o(x, t)$, $t \in (-\infty, \infty)$, for x fixed. Note that the function plotted here is just the "inverse" of the (two-valued) relation plotted in Fig. 5, where the upper curve of that relation is first "fixed up" by removing that part of its domain where it is not one-to-one. The shape of the portion of the curve determined by t_f can be inferred from Fig. 5 or, more formally, from the result $(dt_f/dt)(t)=[d(t_{rm}^{-1})/dt](t) = 1/(dt_{rm}/dt)(t_{rm}^{-1}(t))$ for $t \in [t_{rm}(\bar{t}_o) + |x|/c, \infty)\backslash\{t_\Delta\}$. Finally, if there are several, say $N_\Delta > 1$, intervals of constancy of t_{rm} then there are N_Δ places, t_{Δ_i}, $i = 1,...N_\Delta$, where the map $t \mapsto \overline{T}_o(x, t)$ has a jump discontinuity (if $N_\Delta = 0$ then there are no such places).

Such an expression is given by the unique solution of Eq. (4.58); however, as discussed in Appendix C, we choose not obtain this solution explicitly since it is not needed to proceed to the radiation limit; indeed, the expressions for $\mathbf{E}_a$ and $\mathbf{B}_a$ of Eqs. (4.73) and (4.74) suffice for that purpose. Equations (4.73) and (4.74) thus give, adequately for our purposes, the final representation of the finitely-remote electromagnetic fields in terms of only the given input parameters of the pulse and surface and the spacetime point (x, t) of interest. The only approximations we have made are (1) small spot size: a/ξ, $a/cT \ll 1$; and (2) $v_o/c \ll 1$, which allowed us to neglect terms of order $(v_o/c)^2$ in the radiated fields. Expressions for the customary finitely-remote radiation quantitites may be formed at this point from these fields according to Eqs. (4.16)-(4.20), but they suffer from the same deficiency as the field expressions themselves, namely, the lack of an explicit expression for the small-spot retarded time $\bar{t}'(t_o; \mathbf{x}, t)$. For this reason, we defer treatment of the radiation quantities until after we obtain, in the next section, the radiation limit for the fields.

V. ASYMPTOTIC FIELDS

The finitely-remote fields, given at spacetime point $(\mathbf{x}, t)$ by Eqs. (4.73) and (4.74) in case $t > |\mathbf{x}|/c$ and, more explicitly, by Eq. (4.35) in case $t \leq |\mathbf{x}|/c$, depend upon $\mathbf{x} = |\mathbf{x}|\hat{\mathbf{x}}$ in two different ways; hence, so do the finitely-remote radiation quantities calculated from them *via* Eqs.(4.17)-(4.19). The first dependence on $\mathbf{x}$ is solely through $|\mathbf{x}|$ and is embodied in the circumstance indicated above that the expressions for the fields depend upon the sign of $t - |\mathbf{x}|/c$; this dependence is a causal one and cannot be dispensed with. The second dependence is manifested by first rewriting Eqs. (4.73) and (4.74) respectively as

$$|\mathbf{x}|\mathbf{E}_a(\mathbf{x}, t) = -K'(1/\sin\phi)[\mathbf{i}\,I_{1,3/2}(\mathbf{x}, t) + \mathbf{k}\,I_{0,3/2}(\mathbf{x}, t)] \tag{5.1}$$

and

$$|\mathbf{x}|\mathbf{B}_a(\mathbf{x}, t) = (K'/c)(1/\sin\phi)I_{0,1}(\mathbf{x}, t)\mathbf{j}, \tag{5.2}$$

where $\phi \in (0, \pi/2]$ is the angle between $\mathbf{x} = (\xi, 0, \zeta)$ and $\mathbf{k}$ (see Fig. 4) and

$$\begin{aligned} I_{p,q}(\mathbf{x}, t) \equiv \int_0^{T_o(\mathbf{x},t)} & dt_o G'(\Omega t_o)G(\Omega t_o)\, \Lambda_a^p(t_o; \bar{t}'(t_o; \mathbf{x}, t))\Big\{[1 + \Lambda_a^2(t_o; \bar{t}'(t_o; \mathbf{x}, t))]^{-q} \\ & - 3[V(t_o; \bar{t}'(t_o; \mathbf{x}, t))/c]\Lambda_a(t_o; \bar{t}'(t_o; \mathbf{x}, t))[1 + \Lambda_a^2(t_o; \bar{t}'(t_o; \mathbf{x}, t))]^{-(q+1/2)}\Big\}; \end{aligned} \tag{5.3}$$

and by secondly noting that in general $I_{p,q}(\mathbf{x}, t)$ indeed depends upon $\mathbf{x}$ through both $|\mathbf{x}|$ and $\hat{\mathbf{x}}$. This situation is unsatisfactory since the angular density of a (conceptually idealized) radiation field ought to depend in this second sense only upon direction $\hat{\mathbf{x}}$ and not additionally upon $|\mathbf{x}|$; this second dependence upon $\mathbf{x}$, insofar as it involves $|\mathbf{x}|$, should be eliminated. We accomplish this by defining,

for

$$\hat{\mathbf{x}} = (\sin\phi, 0, \cos\phi), \qquad \phi \in (0, \pi/2] \tag{5.4}$$

and

$$\tau \in (-\infty, \infty), \tag{5.5}$$

the functions

$$\pi_{-}^{E}(\hat{\mathbf{x}}, \tau) = \lim_{|\mathbf{x}| \to \infty} [|\mathbf{x}| \mathbf{E}_{s}(|\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau)] \tag{5.6}$$

and

$$\pi_{-}^{B}(\hat{\mathbf{x}}, \tau) = \lim_{|\mathbf{x}| \to \infty} [|\mathbf{x}| \mathbf{B}_{s}(|\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau)], \tag{5.7}$$

provided the limits exist. If they do, then we define the *asymptotic* instantaneous radiated power angular density at $\mathbf{x}$, $|\mathbf{x}| > 0$, in the direction $\hat{\mathbf{x}}$, as

$$(\delta P/\delta\Psi)_{-}(\mathbf{x}, t) = \varepsilon_{o} c^{2} \hat{\mathbf{x}} \cdot \pi_{-}^{E}(\hat{\mathbf{x}}, t - |\mathbf{x}|/c) \times \pi_{-}^{B}(\hat{\mathbf{x}}, t - |\mathbf{x}|/c), \tag{5.8}$$

as suggested by Eq. (4.17); other asymptotic radiation quantities easily follow. Of course when $\tau \le 0$ then, by Eq. (4.35), the limits in Eqs. (5.6) and (5.7) are trivial:

$$\pi_{-}^{E}(\hat{\mathbf{x}}, \tau) = 0 = \pi_{-}^{B}(\hat{\mathbf{x}}, \tau) \qquad \text{if } \tau \le 0; \tag{5.9}$$

hence we restrict ourselves in the sequel to

$$\tau > 0 \tag{5.10}$$

unless otherwise indicated.

Before presenting the main result, we address the issue, alluded to briefly after Eq. (4.78), of those special pulses -- examples of which were given in Appendix A -- for which the integration range in Eq. (5.3) must be extended, for special values of x and t, beyond $\overline{T}_o(\mathbf{x}, t)$ to $t_s(t - |\mathbf{x}|/c)$. For such pulses and such (x, t) we have, for all $t_o \in [\overline{T}_o(\mathbf{x}, t), t_s(t - |\mathbf{x}|/c)]$, that $Z(t_o; t - |\mathbf{x}|/c) = 0$ so that $\bar{t}'(t_o; \mathbf{x}, t) = t - |\mathbf{x}|/c$; hence $Z(t_o; \bar{t}'(t_o; \mathbf{x}, t)) = 0$ as well so that $\Lambda_*(t_o; \bar{t}'(t_o; \mathbf{x}, t)) = -\zeta/\xi = -\cot\phi$ and $1 + \Lambda_*^2(t_o; \bar{t}(t_o; \mathbf{x}, t)) = 1/\sin^2\phi$. Since also $V(t_o; t - |\mathbf{x}|/c) = -v_o$ then $V(t_o; \bar{t}'(t_o; \mathbf{x}, t)) = -v_o$ as well, hence the additional integral that must be appended for these special cases is simply

$$I^*_{p,q}(\mathbf{x}, t) \equiv \cot^p\phi[\sin^{2q}\phi + 3(v_o/c)\cot\phi \sin^{2q+1}\phi] \int_{\overline{T}_o(\mathbf{x},t)}^{t_s(t - |\mathbf{x}|/c)} dt_o G'(\Omega t_o)G(\Omega t_o). \tag{5.11}$$

Denoting

$$\bar{T}_o^-(\tau) \equiv \begin{cases} \tau, & \text{if } 0 \le \tau \le \bar{t}_o \\ \bar{t}_o, & \text{if } \bar{t}_o < \tau < t_{rm}(\bar{t}_o) \\ t_f(\tau), & \text{if } t_{rm}(\bar{t}_o) \le \tau < \infty, \end{cases} \tag{5.12}$$

so that $\overline{T}_o(\mathbf{x}, t) = \bar{T}_o^-(t - |\mathbf{x}|/c)$ whenever $t - |\mathbf{x}|/c \ge 0$, we then have, using Eq. (4.75),

$$I^*_{p,q}(|\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau) \propto \int_{\bar{T}_o^-(\tau)}^{t_s(\tau)} dt_o G'(\Omega t_o)G(\Omega t_o) = (1/2)\Omega^{-1}[G^2(\Omega t_s(\tau)) - G^2(\Omega \bar{T}_o^-(\tau))] \tag{5.13}$$

so $I^*_{p,q}(|\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau)$ is independent of $|\mathbf{x}|$; hence

$$\lim_{|x| \mapsto \infty} \vec{\Gamma}_{pq}(|x|\hat{x}, |x|/c + \tau) = (1/2)\Omega^{-1}\cot^p\phi[\sin^{2q}\phi + 3(v_o/c)\cot\phi \sin^{2q+1}\phi]$$

$$\times [G^2(\Omega t_s(\tau)) - G^2(\Omega \bar{T}_o^-(\tau))] \tag{5.14}$$

and the existence of the limit is demonstrated for this additional piece. For this reason, we will dispense with further consideration of these special pulses; i.e., in the sequel we will only consider

$$t_o \in [0, \overline{T}_o(x, t)]. \tag{5.15}$$

For such t_o we have from Theorem C.3 that, whenever $\tau = t - |x|/c \geq 0$, then

$$\bar{t}'(t_o; x, t) \in [t_o, t_o + T/G(\Omega t_o)] \tag{5.16}$$

and

$$t - |x|/c \in [t_o, t_o + T/G(\Omega t_o)]. \tag{5.17}$$

Also, for future use, we plot in Fig. 7 the function $\bar{T}_o^-(\tau)$, $\tau \geq 0$, corresponding to the function $\overline{T}_o(x, t)$, $t \geq |x|/c$ of Fig. 6 except that, in consonance with the previous discussion in this paragraph, we take $N_\Delta = 0$ in the latter figure.

The next result establishes the existence of the limits in Eqs. (5.6) and (5.7) when Equations (5.4) and (5.10) govern ϕ and τ. The proof of this theorem is given in Appendix D.

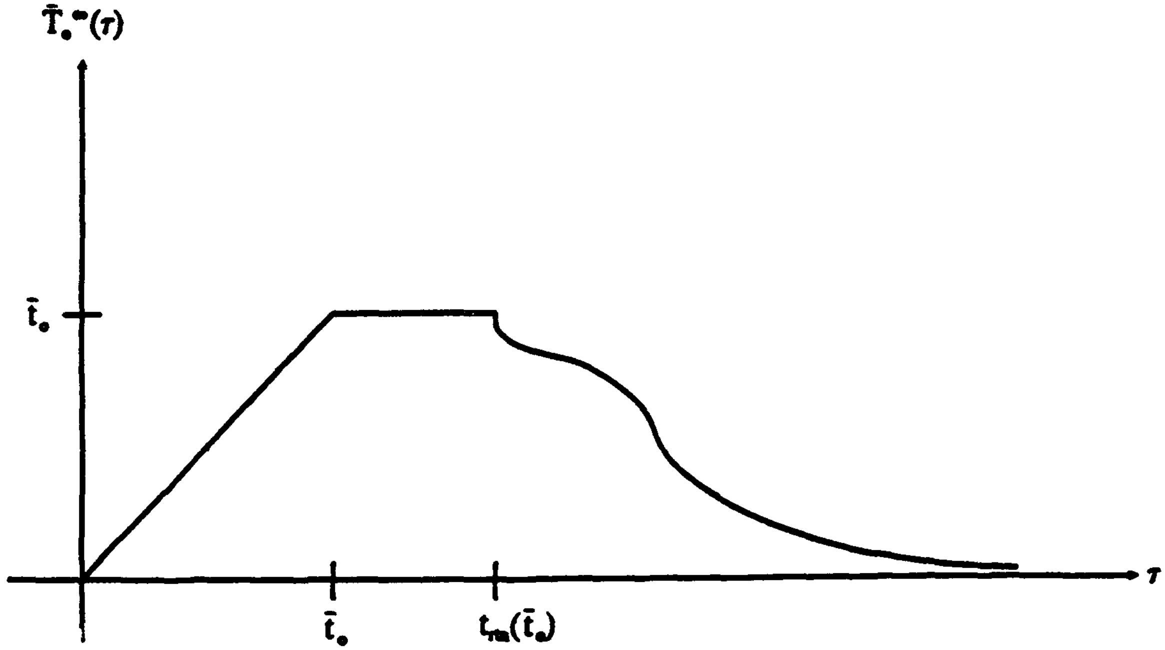


Figure 7. The function $\bar{T}_o^-(\tau)$, $\tau \geq 0$, corresponding to the function $\overline{T}_o(\mathbf{x}, t)$, $t \geq |\mathbf{x}|/c$, of Fig.6, except that here $N_\Delta = 0$.

Theorem 5.1: Let $\hat{\mathbf{x}} = (\sin\phi, 0, \cos\phi)$ with $0 < \phi \le \pi/2$ and $\tau > 0$. Then

$$\lim_{|\mathbf{x}|\to\infty} [\,|\mathbf{x}|\mathbf{E}_a(|\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau)] = (\hat{\mathbf{i}} - \hat{\mathbf{k}}\tan\phi)K'\sin\phi\cos\phi \int_0^{\bar{T}_o^-(\tau)} dt_o G'(\Omega t_o)G(\Omega t_o)$$
$$\times \{1 + 3[V(t_o; \bar{t}'_-(t_o; \phi, \tau))/c]\cos\phi\} \tag{5.18}$$

and

$$\lim_{|\mathbf{x}|\to\infty} [\,|\mathbf{x}|\mathbf{B}_a(|\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau)] = \hat{\mathbf{j}}(K'/c)\sin\phi \int_0^{\bar{T}_o^-(\tau)} dt_o G'(\Omega t_o)G(\Omega t_o)$$
$$\times \{1 + 3[V(t_o; \bar{t}'_-(t_o; \phi, \tau))/c]\cos\phi\}, \tag{5.19}$$

where, if $t_o \neq 0$, then

$$V(t_o; \bar{t}'_-(t_o; \phi, \tau))/c = \begin{cases} (v_o/c)[1 - 2T^{-1}G(\Omega t_o)(\tau - t_o)], & \text{if } \phi = \pi/2 \\ \\ (1/\cos\phi)\left(1 - C_1(\phi)\{1 + [2C_2(\phi)/C_1^2(\phi)]G(\Omega t_o)(\tau - t_o)\}^{1/2}\right), & \text{if } 0 < \phi < \dfrac{\pi}{2} \end{cases} \tag{5.20}$$

with

$$C_1(\phi) = 1 - (v_o/c)\cos\phi \tag{5.21}$$

and

$$C_2(\phi) = 2T^{-1}(v_o/c)\cos\phi\,. \tag{5.22}$$

It is shown in Appendix D that Eq. (5.20) may be approximated, to first order in v_0/c, by

$$V(t_o; \vec{t}'_-(t_o; \phi, \tau))/c = (v_o/c)[1 - 2T^{-1}G(\Omega t_o)(\tau - t_o)], \qquad 0 < \phi \leq \pi/2. \tag{5.23}$$

We may then use this result to write our final expressions for the asymptotic values of π_-^E and π_-^B, namely,

$$\pi_-^E(\hat{x}, \tau) = K' \sin\phi \, K_-(\hat{x}, \tau)(\hat{j} \times \hat{x}) \tag{5.24}$$

and

$$\pi_-^B(\hat{x}, \tau) = (K'/c)\sin\phi \, K_-(\hat{x}, \tau)\hat{j}, \tag{5.25}$$

where

$$K_-(\hat{x}, \tau) = J_-(1, 0; \tau) + 3(v_o/c)\cos\phi \, [J_-(1, 0; \tau) - 2J_-(2, 1; \tau)] \tag{5.26}$$

for

$$J_-(1, 0; \tau) = \int_0^{\bar{T}_o^-} dt_o G'(\Omega t_o)G(\Omega t_o) = (2\pi)^{-1}(\pi/\Omega)G^2(\Omega \bar{T}_o^-(\tau)) \tag{5.27}$$

and

$$\begin{aligned} J_-(2, 1; \tau) &= T^{-1} \int_0^{\bar{T}_o^-(\tau)} dt_o G'(\Omega t_o)G^2(\Omega t_o)(\tau - t_o) \\ &= (3\pi)^{-1}(\pi/\Omega)\{T^{-1}[\tau - \bar{T}_o^-(\tau)]G^3(\Omega \bar{T}_o^-(\tau)) + G_{-,3}(\bar{T}_o^-(\tau))\} \end{aligned} \tag{5.28}$$

with

$$G_{\infty,\mu}(s) \equiv T^{-1}\int_0^s dt_o G^{\mu}(\Omega t_o), \qquad \mu \in \mathbf{N}_o, \quad s \geq 0; \tag{5.29}$$

here $\tau > 0$, $0 < \phi \leq \pi/2$, and π/Ω is the pulse width.

VI. RADIATION QUANTITIES

Having obtained Eqs. (5.24) and (5.25) as characterizing the asymptotic electric and magnetic fields, we are now prepared to obtain the (asymptotic) radiation quantities delineated in Eqs. (4.16) - (4.19), as well as some additional ones. We reiterate that, as per Eq. (5.4), $0 < \phi \leq \pi/2$.

A. Poynting Vector

Guided by Eq. (4.16), we define the asymptotic Poynting vector by

$$\mathbf{S}_{a,-}(\mathbf{x}, t) = \varepsilon_o c^2 |\mathbf{x}|^{-2} \pi_-^E(\hat{\mathbf{x}}, t - |\mathbf{x}|/c) \times \pi_-^B(\hat{\mathbf{x}}, t - |\mathbf{x}|/c). \tag{6.1}$$

From Eqs. (5.9), (5.24), and (5.25) we then have, to first order in v_o/c,

$$\mathbf{S}_{a,-}(\mathbf{x}, t) = \begin{cases} \varepsilon_o c K'^2 |\mathbf{x}|^{-2} \sin^2\phi \, K_{2,-}(\hat{\mathbf{x}}, t - |\mathbf{x}|/c)\hat{\mathbf{x}}, & \text{if } t > |\mathbf{x}|/c \\ 0, & \text{if } t \leq |\mathbf{x}|/c \end{cases} \tag{6.2}$$

where $K_{2,-}(\hat{\mathbf{x}}, \tau)$ is in fact $K_-^2(\hat{\mathbf{x}}, \tau)$ minus its term containing $(v_o/c)^2$; namely

$$K_{2,-}(\hat{\mathbf{x}}, \tau) \equiv J_-^2(1,0;\tau) + 6(v_o/c)\cos\phi \, J_-(1,0;\tau)[J_-(1,0;\tau) - 2\, J_-(2,1;\tau)], \tag{6.3}$$

and neglect of the $(v_o/c)^2$ term in K_-^2 is justified as follows. From Eqs. (5.28) and (5.17) we have

$$0 < J_-(2,1;\tau) \leq T^{-1} \int_0^{\bar{T}_o^-(\tau)} dt_o G'(\Omega t_o) G^2(\Omega t_o)[T/G(\Omega t_o)] = J_-(1,0;\tau); \tag{6.4}$$

since $\tau > 0$ then, by Eq. (5.12), $\bar{T}_o^-(\tau) > 0$ hence $J_-(1,0;\tau) > 0$ and

$$J_-(2,1;\tau)/J_-(1,0;\tau) \leq 1. \tag{6.5}$$

Therefore

$$9\cos^2\phi(v_o/c)^2|J_-(1,0;\tau) - 2\, J_-(2,1;\tau)|^2/J_-^2(1,0;\tau) < 81(v_o/c)^2 \tag{6.6}$$

so that the term in K_-^2 which is second order in v_o/c may be neglected relative to $J_-^2(1,0;\tau)$.

B. Radiated Power

1. Angular Density

From Eqs. (5.8), (5.9), (5.24), and (5.25) or, equivalently, from (see Eq (4.17))

$$(\delta P/\delta\Psi)_-(\mathbf{x},t) = |\mathbf{x}|^2\hat{\mathbf{x}} \cdot \mathbf{S}_{a,-}(\mathbf{x},t), \tag{6.7}$$

we find the asymptotic instantaneous radiated power per unit solid angle (Ψ) in the direction

$$\hat{\mathbf{x}} = \mathbf{i}\sin\phi + \mathbf{k}\cos\phi \neq \mathbf{k}, \tag{6.8}$$

for $|\mathbf{x}| \gg 0$, to be

$$(\delta P/\delta\Psi)_-(\mathbf{x},t) = \begin{cases} \varepsilon_o c K'^2 \sin^2\phi\, K_{2,-}(\hat{\mathbf{x}}, t - |\mathbf{x}|/c), & \text{if } t > |\mathbf{x}|/c \\ 0, & \text{if } t \leq |\mathbf{x}|/c \end{cases} \tag{6.9}$$

(to first order in v_o/c). We also find using Fig. 5 and Eqs. (6.3) and (5.27) that the peak (in time) radiated power per steradian at any $\phi \in (0, \pi/2]$ is given approximately (neglecting the term of order v_o/c) by

$$(\delta P/\delta\Psi)_-^{peak}(\hat{x}) = \varepsilon_o c K'^2 \sin^2\phi (2\pi)^{-2}(\pi/\Omega)^2 G^4(\Omega\bar{t}_o) \tag{6.10}$$

at time $t = |\mathbf{x}|/c + \bar{t}_o$.

2. Angularly Integrated Power

Noting that ϕ is in fact the polar angle measured from the (positive) z-axis, we compute the integral of $(\delta P/\delta\Psi)_-(\mathbf{x}, t)$ over the forward hemisphere $\xi^2 + \eta^2 + \zeta^2 = |\mathbf{x}|^2 > 0$, $\zeta \geq 0$, as (since $(\delta P/\delta\Psi)_-$ is only defined for $\phi > 0$)

$$P_-(|\mathbf{x}|, t) = \lim_{\phi \to 0^+} [2\pi \int_{\phi}^{\pi/2} d\phi' \; \sin\phi' (\delta P/\delta\Psi)_-(\mathbf{x}, t)]. \tag{6.11}$$

We find

$$P_-(|\mathbf{x}|, t) = \begin{cases} \varepsilon_o c K'^2 K_{2,-}^{I}(t - |\mathbf{x}|/c), & \text{if } t > |\mathbf{x}|/c \\ 0, & \text{if } t \leq |\mathbf{x}|/c \end{cases} \tag{6.12}$$

(to first order in v_o/c) where

$$K_{2,-}^{I}(\tau) = \pi\{(4/3)J_-^2(1,0;\tau) + 3(v_o/c)J_-(1,0;\tau)[J_-(1,0;\tau) - 2J_-((2,1;\tau)]\}. \tag{6.13}$$

C. Radiated Energy

1. Angular Density

From Eqs. (4.18) and (6.9), the asymptotic total radiated energy per unit solid angle (to first order in v_o/c) is given by

$$
\begin{aligned}
(\delta W/\delta\Psi)_-(\hat{k}) &= \varepsilon_o c K'^2 \sin^2\phi \int_0^\infty d\tau K_{2,-}(\hat{k},\tau) \\
&= \varepsilon_o c K'^2 \sin^2\phi \{[1 + 6(v_o/c)\cos\phi] \int_0^\infty d\tau J_-^2(1,0;\tau) \\
&\quad - 12(v_o/c)\cos\phi \int_0^\infty d\tau J_-(1,0;\tau) J_-(2,1;\tau)\}.
\end{aligned} \tag{6.14}
$$

The first integral in the above equation may be evaluated by writing

$$
\int_0^\infty d\tau J_-^2(1,0;\tau) = \int_0^\infty d\tau \int_0^{\bar{t}_o^-(\tau)} ds_o G'(\Omega s_o) G(\Omega s_o) \int_0^{\bar{t}_o^-(\tau)} dt_o G'(\Omega t_o) G(\Omega t_o) \tag{6.15}
$$

and then observing that the integration domain for the above triple integral is that illustrated in Fig. 8, so that a change of integration order gives (using Eq. (3.24) for $t_{rm}(\cdot)$)

$$
\begin{aligned}
\int_0^\infty d\tau J_-^2(1,0;\tau) &= \int_0^{\bar{t}_o} ds_o G'(\Omega s_o) G(\Omega s_o) \int_0^{\bar{t}_o} dt_o G'(\Omega t_o) G(\Omega t_o) \int_{\max\{s_o, t_o\}}^{\min\{t_{rm}(s_o),\, t_{rm}(t_o)\}} d\tau \\
&= \int_0^{\bar{t}_o} ds_o G'(\Omega s_o) G(\Omega s_o) \int_0^{\bar{t}_o} dt_o G'(\Omega t_o) G(\Omega t_o) (\min\{t_{rm}(s_o), t_{rm}(t_o)\} - \max\{s_o, t_o\}) \\
&= \{ \int_0^{\bar{t}_o} ds_o G'(\Omega s_o) G(\Omega s_o) \int_0^{s_o} dt_o G'(\Omega t_o) G(\Omega t_o) [T/G(\Omega s_o)] \\
&\quad + \int_0^{\bar{t}_o} ds_o G'(\Omega s_o) G(\Omega s_o) \int_{s_o}^{\bar{t}_o} dt_o G'(\Omega t_o) G(\Omega t_o) [T/G(\Omega t_o)]\} \\
&= (1/3)\Omega^{-2} T G^3(\Omega \bar{t}_o).
\end{aligned} \tag{6.16}
$$

Similarly,

$$
\int_0^\infty d\tau\, J_-(1,0;\tau) J_-(2,1;\tau) = \frac{5}{36}\, \Omega^{-2} T G^3(\Omega \bar{t}_o) + \frac{1}{3}\, \Omega^{-2} T [G(\Omega \bar{t}_o) G_{-,3}(\bar{t}_o) - G_{-,4}(\bar{t}_o)] \tag{6.17}
$$

where $G_{-,\mu}(\bar{t}_o)$ is given by Eq. (5.29) with $s = \bar{t}_o$. Hence we have

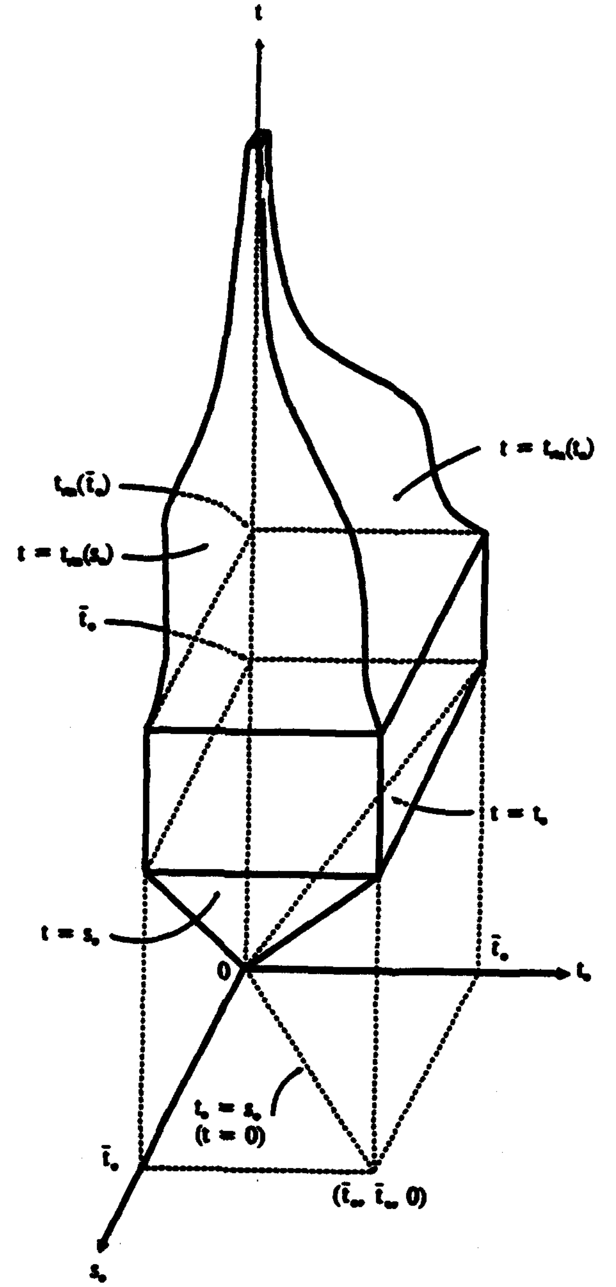


Figure 8. The integration domain for the triple integrals of Eqs.(6.15) and (6.16), corresponding to the $\bar{T}_o^-(\tau)$, of Fig. 7. Given $(s_o, t_o) \in [0, \bar{t}_o]^2$, the lower limit for the t integration is s_o if $t_o \le s_o$ or it is t_o if $s_o \le t_o$; i.e., it is $\max\{s_o, t_o\}$. Likewise, the upper limit is $\min\{t_{rm}(s_o), t_{rm}(t_o)\}$.

$$(\delta W/\delta\Psi)_\infty(\hat{x}) = \tfrac{1}{3}\pi^{-2}\varepsilon_o c K'^2 T(\pi/\Omega)^2 \sin^2\phi \Big(G^3(\Omega\bar{t}_o) + (v_o/c)\cos\phi\{G^3(\Omega\bar{t}_o) - 12[G(\Omega\bar{t}_o)G_{\infty,3}(\bar{t}_o) - G_{\infty,4}(\bar{t}_o)]\}\Big). \qquad (6.18)$$

2. Total Energy

The asymptotic total radiated energy is just the angularly integrated asymptotic (energy) angular density, namely

$$W_\infty = \lim_{\phi\to 0^+}\Big[2\pi\int_\phi^{\pi/2} d\phi' \sin\phi' (\delta W/\delta\Psi)_\infty(\hat{x})\Big]$$

$$= \tfrac{1}{3}\pi^{-1}\varepsilon_o c K'^2 T(\pi/\Omega)^2\Big(\tfrac{4}{3}G^3(\Omega\bar{t}_o) + \tfrac{1}{2}(v_o/c)\{G^3(\Omega\bar{t}_o) - 12[G(\Omega\bar{t}_o)G_{\infty,3}(\bar{t}_o) - G_{\infty,4}(\bar{t}_o)]\}\Big). \qquad (6.19)$$

3. Efficiency

Noting that the photon pulse input energy for photons of frequency ν is (using Eq. (2.7))

$$W_{in} = h\nu\pi a^2 A\int_0^{\bar{t}_o} f(t)dt = h\nu(\pi/\Omega)a^2 A\Omega N(\Omega)G(\Omega\bar{t}_o), \qquad (6.20)$$

which will be less than the total energy in the photon pulse if $\bar{t}_o < \pi/\Omega$, we may calculate an asymptotic radiation efficiency, using K' from Eq. (4.78) and T from Eq. (3.25), as

$$e_\infty^{rad} \equiv W_\infty/W_{in} = (1/24\pi^2)(v_o/c)m_e(e^2/\varepsilon_o m_e c)^2(h\nu)^{-1}(\pi/\Omega)^2[\Omega N(\Omega)]^2 a^2(AY)^2 Y$$
$$\times\Big[\tfrac{4}{3}G^2(\Omega\bar{t}_o) + \tfrac{1}{2}(v_o/c)\Big(G^2(\Omega\bar{t}_o) - 12\{G_{\infty,3}(\bar{t}_o) - [G_{\infty,4}(\bar{t}_o)/G(\Omega\bar{t}_o)]\}\Big)\Big]. \qquad (6.21)$$

Note that $(eAY)^2$ is the square of the emission current density and, from Eq. (2.6),

$$\Omega N(\Omega) = \int_0^{\pi} g(s)ds. \tag{6.22}$$

Also, since for $\mu' \geq \mu \geq 0$ we have

$$|G_{\infty,\mu'}(\bar{t}_o)/G^{\mu}(\Omega\bar{t}_o)| \leq T^{-1}\int_0^{\bar{t}_o} dt_o[G^{\mu}(\Omega t_o)/G^{\mu}(\Omega\bar{t}_o)]G^{\mu'-\mu}(\Omega t_o) < T^{-1}\int_0^{\bar{t}_o} dt_o = \bar{t}_o/T \tag{6.23}$$

then (noting $G(\Omega\bar{t}_o) > 0$ since $\bar{t}_o > 0$ by Theorem 3.2(i))

$$\tfrac{1}{2}|G^2(\Omega\bar{t}_o) - 12\{G_{\infty,3}(\bar{t}_o) - [G_{\infty,4}(\bar{t}_o)/G(\Omega\bar{t}_o)]\}| \,/\, \tfrac{4}{3}\, G^2(\Omega\bar{t}_o) \leq \tfrac{3}{8}[1 + 24(\bar{t}_o/T)] \tag{6.24}$$

so that the v_o/c term inside the large brackets in Eq. (6.21) may be ignored whenever $v_o/c \cdot (3/8)[1 + 24(\bar{t}_o/T)] \lesssim 0.05$ (0.05 is arbitrary but about right), i.e., whenever $\bar{t}_o/T \lesssim 1/2$. A sufficient condition for this is

$$1 \geq 2(\pi/\Omega)T^{-1} = \pi\Omega N(\Omega)(\omega_p/\iota\lambda)^2 \tag{6.25}$$

and in that case we have

$$e_\infty^{rad} = (1/18\pi^2)(v_o/c)m_e(e^2/\varepsilon_o m_e c)^2(h\nu)^{-1}(\pi/\Omega)^2[\Omega N(\Omega)]^2 a^2(AY)^2 Y G^2(\Omega\bar{t}_o). \tag{6.26}$$

D. Spectral Intensity

1. Angular Density

We may calculate the asymptotic spectral intensity angular density $(\delta^2 I/\delta\Psi\,\delta\omega)_\infty(\hat{x}, \omega)$,

specified unambiguously by (see Eq. (4.21))

$$(\delta W/\delta\Psi)_-(\hat{x}) = \int_0^{\infty} d\omega\,(\delta^2 I/\delta\Psi\,\delta\omega)_-(\hat{x}, \omega), \tag{6.27}$$

from (see Eq. (4.19))

$$(\delta^2 I/\delta\Psi\,\delta\omega)_-(\hat{x}, \omega) = 2\varepsilon_o c^2\, Re[\hat{x} \cdot \pi^{E}_{-\wedge}(\hat{x}, \omega) \times (\pi^{B}_{-\wedge})^*(\hat{x}, \omega)] \tag{6.28}$$

(where $\wedge$ denotes Fourier transform and $*$ denotes complex conjugate). That this specification of the asymptotic of spectral density is indeed the correct one follows from

$$\begin{aligned}
(\delta W/\delta\Psi)_-(\hat{x}) &= \int_{-\infty}^{\infty} d\tau (\delta P/\delta\Psi)_-(\hat{x}, \tau) \\
&= \varepsilon_o c^2 \hat{x} \cdot \int_{-\infty}^{\infty} d\tau\; \pi^{E}_{-}(\hat{x}, \tau) \times \pi^{B}_{-}(\hat{x}, \tau) \\
&= \varepsilon_o c^2 \hat{x} \cdot \int_{-\infty}^{\infty} d\tau [(2\pi)^{-1/2} \int_{-\infty}^{\infty} d\omega\; \pi^{E}_{-\wedge}(\hat{x}, \omega) e^{-i\omega\tau} \times (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\omega'\, \pi^{B}_{-\wedge}(\hat{x}, \omega') e^{-i\omega'\tau}] \\
&= \varepsilon_o c^2 \hat{x} \cdot \int_{-\infty}^{\infty} d\omega\; \pi^{E}_{-\wedge}(\hat{x}, \omega) \times (\pi^{B}_{-\wedge})^*(\hat{x}, \omega) \\
&= \int_0^{\infty} d\omega\, [2\varepsilon_o c^2 \hat{x} \cdot \pi^{E}_{-\wedge}(\hat{x}, \omega) \times (\pi^{B}_{-\wedge})^*(\hat{x}, \omega)].
\end{aligned} \tag{6.29}$$

To compute the spectral intensity via Eq. (6.28) we first supplement the definition of $K_-(\hat{x}, \tau)$ for $\tau > 0$ in Eq. (5.26) with

$$K_-(\hat{x}, \tau) = 0 \text{ if } \tau \le 0 \tag{6.30}$$

so that Eqs. (5.24) and (5.25) hold even when $\tau \le 0$ and are in agreement with Eq. (5.9). Now

$$\pi^{E}_{-\Lambda}(\hat{\mathbf{x}}, \omega) = (\hat{\mathbf{j}} \times \hat{\mathbf{x}})\, K' \sin\phi\, K_{-\Lambda}(\hat{\mathbf{x}}, \omega) \tag{6.31}$$

and

$$\pi^{B}_{-\Lambda}(\hat{\mathbf{x}}, \omega) = \mathbf{j}(K'/c)\sin\phi\;\, K_{-\Lambda}(\hat{\mathbf{x}}, \omega) \tag{6.32}$$

with

$$\begin{aligned} K_{-\Lambda}(\hat{\mathbf{x}}, \omega) &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\tau\; e^{i\omega\tau} K_{-}(\hat{\mathbf{x}}, \tau) = (2\pi)^{-1/2} \int_{0}^{\infty} d\tau\; e^{i\omega\tau} K_{-}(\hat{\mathbf{x}}, \tau) \\ &= (2\pi)^{-1/2}\{[1+3(v_o/c)\cos\phi] \int_{0}^{\infty} d\tau\; e^{i\omega\tau} J_{-}(1, 0; \tau) - 6(v_o/c)\cos\phi \int_{0}^{\infty} d\tau\; e^{i\omega\tau} J_{-}(2, 1; \tau)\}. \end{aligned} \tag{6.33}$$

Using $\hat{\mathbf{x}}$ from Eq. (6.8) we then have

$$(\delta^2 I/\delta\Psi\, \delta\omega)_{-}(\hat{\mathbf{x}}, \omega) = 2\varepsilon_o c K'^2 \sin^2\phi\, |K_{-\Lambda}(\hat{\mathbf{x}}, \omega)|^2. \tag{6.34}$$

The integrals in $K_{-\Lambda}$ may be performed as earlier (see Eqs. (6.15) - (6.17)):

$$\begin{aligned} \int_{0}^{\infty} d\tau e^{i\omega\tau} J_{-}(1, 0; \tau) &= \int_{0}^{\infty} d\tau e^{i\omega\tau} \int_{0}^{\bar{T}_o^{-}(\tau)} dt_o G'(\Omega t_o) G(\Omega t_o) \\ &= \int_{0}^{T_o} dt_o G'(\Omega t_o) G(\Omega t_o) \int_{t_o}^{t_{m}(t_o)} d\tau e^{i\omega\tau} \\ &= -i\omega^{-1} \int_{0}^{T_o} dt_o G'(\Omega t_o) G(\Omega t_o) e^{i\omega t_o} [e^{i\omega T G(\Omega t_o)} - 1], \end{aligned} \tag{6.35}$$

where this integral is proper since both

$$\lim_{t_o \to 0^+} G(\Omega t_o) e^{i\omega T G(\Omega t_o)} = 0 \tag{6.36}$$

and $G'(\Omega t_o) = g(\Omega t_o)/\Omega N(\Omega)$ hold; and, similarly,

$$\int_0^\infty d\tau e^{i\omega\tau} J_-(2,1;\tau) = -i\omega^{-1} T^{-1} \int_0^{\tau_o} dt_o G'(\Omega t_o) G(\Omega t_o) t_o \, e^{i\omega t_o} [e^{i\omega T G(\Omega t_o)} - 1]. \tag{6.37}$$

Hence we finally have

$$(\delta^2 I/\delta\Psi\, \delta\omega)_-(\hat{x}, \omega) = \pi^{-3} \varepsilon_o c K'^2 (\pi/\Omega)^2 (\Omega/\omega)^2 \sin^2\phi$$
$$\times \left| \int_0^{\tau_o} dt_o G'(\Omega t_o) G(\Omega t_o) e^{i\omega t_o} [e^{i\omega T G(\Omega t_o)} - 1] \{1 + 3(v_o/c)\cos\phi[1 - 2(t_o/T)]\} \right|^2. \tag{6.38}$$

2. Angularly Integrated Spectral Intensity

The general expression for

$$(\delta I/\delta\omega)_-(\omega) = \lim_{\phi \to 0^+} [2\pi \int_\phi^{\pi/2} d\phi' \sin\phi' (\delta^2 I/\delta\Psi\, \delta\omega)(\hat{x}, \omega)], \tag{6.39}$$

while straightforward to compute, is cumbersome since $\cos\phi$ occurs inside $|K_{-\Lambda}(\hat{x}, \omega)|^2$; we will not display it here.

VII. EXAMPLES

In this final section, we present illustrations of our general formalism for the following pulses: constant, linear ramp, triangular, parabolic, and $\sin^2$. While we have complete, detailed results for all these pulses, we present such detail here only for the constant pulse; for the others, we discuss merely those of their features which differ significantly from those of the constant pulse.

A. Constant

A constant (or "flattop") pulse is given by

$$\hat{I}(t) = A, \qquad t \in [0, \pi/\Omega] \tag{7.1}$$

for arbitrary $A > 0$. We then have, for $s = \Omega/t \in [0, \pi]$,

$$g(s) = 1, \qquad G(\Omega t) = t/(\pi/\Omega), \qquad G'(\Omega t) = 1/\pi, \qquad \Omega N(\Omega) = \pi; \tag{7.2}$$

and

$$T = 2/(\pi/\Omega)\omega_p^2 = 2\kappa^{-1}(v_o/c)/(\pi/\Omega)AY \tag{7.3}$$

where

$$\kappa \equiv e^2/\varepsilon_o m_e c. \tag{7.4}$$

Now for $t_o \in [0, \pi/\Omega]$,

$$t_{rm}(t_o) = (\pi/\Omega)\{[t_o/(\pi/\Omega)] + [T/(\pi/\Omega)][t_o/(\pi/\Omega)]^{-1}\} = t_o + (2/\omega_p^2)t_o^{-1} \tag{7.5}$$

so t_{rm} has unique local minimum at

$$t_{o,min}(T) = (\pi/\Omega)[T/(\pi/\Omega)]^{1/2} = \sqrt{2}/\omega_p = \sqrt{2}\,\kappa^{-1/2}(v_o/c)^{1/2}/(AY)^{1/2}. \tag{7.6}$$

We then see that $t_{o,min}(T) < \pi/\Omega$ iff $T < \pi/\Omega$ so

$$\bar{t}_o = \begin{cases} t_{o,min}(T), & \text{if S} \\ \pi/\Omega, & \text{if W} \end{cases} \tag{7.7}$$

where, observing that $\bar{t}_o$ depends upon ω_p (or T) which in turn varies directly as $A^{1/2}$ (or inversely as A), we have distinguished "strong pulse" (S) and "weak pulse" (W) cases as follows:

$$\begin{aligned} &\text{S:}\quad \omega_p/\Omega > \sqrt{2}/\pi \text{ or } T < \pi/\Omega \\ &\text{W:}\quad \omega_p/\Omega \le \sqrt{2}/\pi \text{ or } T \ge \pi/\Omega . \end{aligned} \tag{7.8}$$

The situation is illustrated in Fig. 9. Continuing, we find

$$t_{rm}(\bar{t}_o) = \begin{cases} 2t_{o,min}(T), & \text{if S} \\ \pi/\Omega + T, & \text{if W} \end{cases} \tag{7.9}$$

and, for $t \ge t_{rm}(\bar{t}_o)$,

$$t_{rm}^{-1}(t) = (1/2)\{t - [t^2 - 4(\pi/\Omega)T]^{1/2}\} = t_I(t) = t_S(t). \tag{7.10}$$

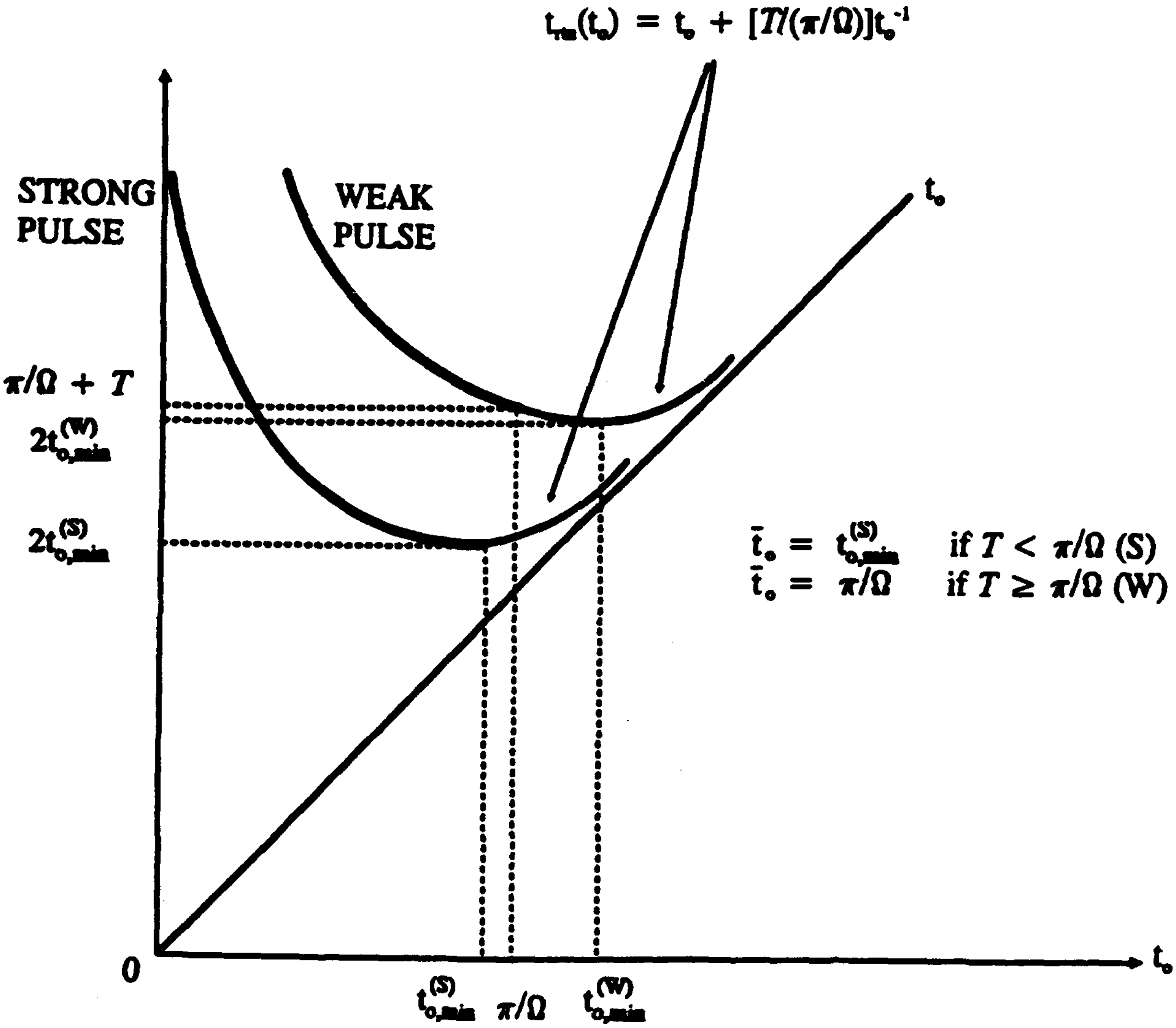


Figure 9. The strong (S) and weak (W) pulse cases for the flattop pulse of Section VII.A.

Then $\bar{T}_o^-$ is explicitly specified for the strong and weak cases by using Eqs. (7.7) and (7.10) in Eq. (5.12); further

$$J_-(1,0;\tau) = (2\pi)^{-1}(\pi/\Omega)[\bar{T}_o^-(\tau)/(\pi/\Omega)]^2, \tag{7.11}$$

$$J_-(2,1;\tau) = (3\pi)^{-1}(\pi/\Omega)[\bar{T}_o^-(\tau)/(\pi/\Omega)]^3 T^{-1}[\tau - \tfrac{3}{4}\bar{T}_o^-(\tau)], \tag{7.12}$$

and

$$K_{2,-}(\hat{\mathbf{x}},\tau) = (2\pi)^{-2}(\pi/\Omega)^2[\bar{T}_o^-(\tau)/(\pi/\Omega)]^4\Big(1 + 6(v_o/c)\cos\phi\{1 - \tfrac{4}{3}[\bar{T}_o^-(\tau)/(\pi/\Omega)]T^{-1}[\tau - \tfrac{3}{4}\bar{T}_o^-(\tau)]\}\Big); \tag{7.13}$$

also,

$$K' = (1/4)\pi\kappa^2(m_e/e)(\pi/\Omega)a^2(AY)^2 \tag{7.14}$$

and

$$\varepsilon_o c K'^2 = (1/16)\pi^2\kappa^3 m_e(\pi/\Omega)^2 a^4(AY)^4. \tag{7.15}$$

We are now prepared to display the radiation quantities. If $t > |\mathbf{x}|/c$ and $\tau = t - |\mathbf{x}|/c$ then the radiated power angular density is

$$\begin{aligned}(\delta P/\delta\Psi)_-(\mathbf{x},t) &= \tfrac{1}{64}\kappa^3 m_e \sin^2\phi\, a^4(AY)^4[\bar{T}_o^-(\tau)]^4\Big(1 + 6(v_o/c)\cos\phi\{1 - \tfrac{4}{3}\bar{T}_o^-(\tau)[\tau - \tfrac{3}{4}\bar{T}_o^-(\tau)]/(\pi/\Omega)T\}\Big)\\ &= \tfrac{1}{64}\kappa^3 m_e \sin^2\phi\, a^4(AY)^4[\bar{T}_o^-(\tau)]^4\{1 - 4\kappa\cos\phi(AY)\bar{T}_o^-(\tau)[\tau - \tfrac{3}{4}\bar{T}_o^-(\tau)] + 6(v_o/c)\cos\phi\}\end{aligned} \tag{7.16}$$

where we have used Eq. (7.3) to obtain the second equality in Eq. (7.16). The approximate peak (in time) power (gotten by neglecting the v_o/c term in the first line of Eq. (7.16)) is

$$(\delta P/\delta\Psi)_{-}^{\text{peak}}(\hat{x}) \approx (1/64)\kappa^3 m_e \sin^2\phi\, a^4(AY)^4(\pi/\Omega)^4 \times \begin{cases} [T/(\pi/\Omega)]^2, & \text{if S} \\ 1, & \text{if W}; \end{cases} \tag{7.17}$$

note that in the strong case this may be rewritten, using Eq. (7.3), as

$$(\delta P/\delta\Psi)_{-}^{\text{peak}}(\hat{x}) = (1/16)\kappa m_e \sin^2\phi\ (v_o/c)^2 a^4(AY)^2 \qquad \text{(S)}. \tag{7.18}$$

The angularly integrated power and approximate peak power are, respectively,

$$P_{-}(|\mathbf{x}|,t) = (1/64)\pi\kappa^3 m_e a^4(AY)^4[\bar{T}_o^-(\tau)]^4\left(\tfrac{4}{3} + 3(v_o/c)\{1 - \tfrac{4}{3}[\bar{T}_o^-(\tau)/(\pi/\Omega)]T^{-1}[\tau - \tfrac{3}{4}\bar{T}_o^-(\tau)]\}\right) \tag{7.19}$$

and

$$P_{-}^{\text{peak}}(|\mathbf{x}|) \approx (1/48)\pi\kappa^3 m_e a^4(AY)^4(\pi/\Omega)^4 \times \begin{cases} [T/\pi/\Omega)]^2, & \text{if S} \\ 1, & \text{if W} \end{cases} \tag{7.20}$$

where, in the strong case, the last may be rewritten as

$$P_{-}^{\text{peak}}(|\mathbf{x}|) \approx (1/12)\pi\kappa m_e (v_o/c)^2 a^4(AY)^2 \qquad \text{(S)}. \tag{7.21}$$

The total radiated energy angular density is

$$
\begin{aligned}
(\delta W/\delta\Psi)_{\infty}(\hat{k}) &= \tfrac{1}{48}\kappa^3 m_e \sin^2\phi\, a^4 (AY)^4(\pi/\Omega)T\,\bar{t}_o^3\{1 + (v_o/c)\cos\phi[1 - \tfrac{3}{5}\bar{t}_o^2/(\pi/\Omega)T]\} \\
&= \tfrac{1}{24}\kappa^2 m_e \sin^2\phi (v_o/c) a^4 (AY)^3\,\bar{t}_o^3\{1 - \tfrac{3}{10}\kappa\cos\phi(AY)\bar{t}_o^2 + (v_o/c)\cos\phi] \\
&= \tfrac{1}{24} m_e \sin^2\phi\, a^4 \\
&\quad \times \begin{cases} 2^{3/2}\kappa^{1/2}(v_o/c)^{5/2}(AY)^{3/2}[1 + \tfrac{2}{5}(v_o/c)\cos\phi], & \text{if S} \\ \kappa^2 (v_o/c)(AY)^3(\pi/\Omega)^3[1 - \tfrac{3}{10}\kappa\cos\phi(\pi/\Omega)^2 AY + (v_o/c)\cos\phi], & \text{if W} \end{cases}
\end{aligned} \tag{7.22}
$$

and the total radiated energy is

$$
\begin{aligned}
W_{\infty} &= \tfrac{1}{24}\pi\kappa^2 m_e (v_o/c) a^4 (AY)^3\,\bar{t}_o^3[\tfrac{4}{3} - \tfrac{3}{20}\kappa AY\,\bar{t}_o^2 + \tfrac{1}{2}(v_o/c)] \\
&= \tfrac{1}{24}\pi m_e a^4 \times \begin{cases} 2^{3/2}\kappa^{1/2}(v_o/c)^{5/2}(AY)^{3/2}[\tfrac{4}{3} + \tfrac{1}{5}(v_o/c)], & \text{if S} \\ \kappa^2(v_o/c)(AY)^3(\pi/\Omega)^3[\tfrac{4}{3} - \tfrac{3}{20}\kappa(\pi/\Omega)^2 AY + \tfrac{1}{2}(v_o/c)], & \text{if W.} \end{cases}
\end{aligned} \tag{7.23}
$$

To compute the radiation efficiency we note that

$$
W_{in} = \pi h\nu a^2 A\bar{t}_o \tag{7.24}
$$

so

$$
\begin{aligned}
e_{\infty}^{rad} &= \tfrac{1}{24}\kappa^2 m_e (v_o/c)(h\nu)^{-1} a^2 (AY)^2 Y\,\bar{t}_o^2[\tfrac{4}{3} - \tfrac{3}{20}\kappa AY\,\bar{t}_o^2 + \tfrac{1}{2}(v_o/c)] \\
&= \tfrac{1}{24} m_e (h\nu)^{-1} a^2 \times \begin{cases} 2\kappa(v_o/c)^2(AY)Y[\tfrac{4}{3} + \tfrac{1}{5}(v_o/c)], & \text{if S} \\ \kappa^2(v_o/c)(AY)^2 Y(\pi/\Omega)^2[\tfrac{4}{3} - \tfrac{3}{20}\kappa(\pi/\Omega)^2 AY + \tfrac{1}{2}(v_o/c)], & \text{if W .} \end{cases}
\end{aligned} \tag{7.25}
$$

Finally, the spectral intensity is given by

$$(\delta^2 I/\delta\Psi\,\delta\omega)_-(\hat{x},\omega) = (1/16)\pi^{-1}\kappa^3 m_e \sin^2\phi\; a^4(AY)^4[M(\omega)/\omega^3]^2 \tag{7.26}$$

where

$$\begin{aligned} M(\omega) = &\Big[C_1(\omega) - (\cos\omega\bar{t}_o + \omega\bar{t}_o\sin\omega\bar{t}_o - 1) \\ &+ 3(v_o/c)\cos\phi\Big(C_1(\omega) - (\cos\omega\bar{t}_o + \omega\bar{t}_o\sin\omega\bar{t}_o - 1) - \pi(\frac{\omega_p}{\Omega})(\frac{\omega_p}{\omega})\{C_2(\omega) \\ &- 2\omega\bar{t}_o\cos\omega\bar{t}_o - [(\omega\bar{t}_o)^2 - 2]\sin\omega\bar{t}_o\}\Big)\Big]^2 \\ &+ \Big[S_1(\omega) - (\sin\omega\bar{t}_o - \omega\bar{t}_o\cos\omega\bar{t}_o) \\ &+ 3\,(v_o/c)\cos\phi\Big(S_1\omega) - (\sin\omega\bar{t}_o - \omega\bar{t}_o\cos\omega\bar{t}_o) - \pi(\frac{\omega_p}{\Omega})(\frac{\omega_p}{\omega})\{S_2(\omega) \\ &- 2\omega\bar{t}_o\sin\omega\bar{t}_o - [(\omega\bar{t}_o)^2 - 2](1 - \cos\omega\bar{t}_o)\}\Big)\Big]^2 \end{aligned} \tag{7.27}$$

for

$$C_q(\omega) \equiv (\sqrt{2}\frac{\omega}{\omega_p})^{q+1}\int_0^{(\omega_p/\sqrt{2})\bar{t}_o} dy\;y^q\cos[\sqrt{2}\frac{\omega}{\omega_p}(y + \frac{1}{y})] \qquad (q = 1, 2) \tag{7.28}$$

and

$$S_q(\omega) \equiv (\sqrt{2}\frac{\omega}{\omega_p})^{q+1}\int_0^{(\omega_p/\sqrt{2})\bar{t}_o} dy\;y^q\sin[\sqrt{2}\frac{\omega}{\omega_p}(y + \frac{1}{y})] \qquad (q = 1, 2). \tag{7.29}$$

Note that in the strong case,

$$\omega\bar{t}_o = \sqrt{2}\frac{\omega}{\omega_p} \qquad \text{and} \qquad (\omega_p/\sqrt{2})\bar{t}_o = 1 \qquad\qquad (S) \tag{7.30}$$

while in the weak case,

$$\omega \bar{t}_o = \pi \frac{\omega}{\Omega} \qquad \text{and} \qquad (\omega_p/\sqrt{2})\bar{t}_o = \frac{\pi}{\sqrt{2}} \frac{\omega_p}{\Omega} \qquad \text{(W)}. \tag{7.31}$$

B. Linear Ramp

The linear ramp

$$\hat{f}(t) = A[t/(\pi/\Omega)], \qquad t \in [0, \pi/\Omega], \qquad A > 0 \tag{7.32}$$

yields results very similar to those for the constant pulse, with the strong pulse regime resulting when $T < (1/2)\,(\pi/\Omega)$ and the weak pulse resulting when $T \geq (1/2)(\pi/\Omega)$.

C. Triangular

For the triangular pulse

$$\hat{f}(t) = 2A \times \begin{cases} t/(\pi/\Omega), & \text{if } 0 \leq t \leq (1/2)(\pi/\Omega) \\ 1 - [t/(\pi/\Omega)], & \text{if } (1/2)(\pi/\Omega) \leq t \leq \pi/\Omega \end{cases} \tag{7.33}$$

$(A > 0)$ we find

$$T = 4/(\pi/\Omega)\omega_p^2 = 4\kappa^{-1}(v_o/c)/(\pi/\Omega)AY \tag{7.34}$$

and

$$t_m(t_o) = (\pi/\Omega) \times \begin{cases} [(t_o/(\pi/\Omega)] + (1/2)[T/(\pi/\Omega)][t_o/(\pi/\Omega)]^{-2}, & \text{if } 0 \leq t_o \leq (1/2)(\pi/\Omega) \\ [t_o/(\pi/\Omega)] + [T/(\pi/\Omega)]\left(1 - 2\{1 - [t_o/(\pi/\Omega)]\}^2\right)^{-1}, & \text{if } (1/2)\,(\pi/\Omega) \leq t_o \leq \pi/\Omega. \end{cases} \tag{7.35}$$

Additionally,

$$
(dt_{rm}/dt_o)(t_o) = \begin{cases} 1 - [T/(\pi/\Omega)][t_o/(\pi/\Omega)]^{-3}, & \text{if } 0 \le t_o \le (1/2)(\pi/\Omega) \\ 1 - 4[T/(\pi/\Omega)]\{1 - [t_o/(\pi/\Omega)]\}\left(1 - 2\{1 - [t_o/(\pi/\Omega)]\}^2\right)^{-2}, & \text{if } (1/2)(\pi/\Omega) \le t_o \le \pi/\Omega \end{cases} \tag{7.36}
$$

and it is easy to check that $(d^2t_{rm}/dt^2)(t_o) > 0$, unless $t_o = (1/2)(\pi/\Omega)$ where this second derivative does not exist; since

$$
(dt_{rm}/dt_o)(\pi/\Omega) = 1 > 0 \tag{7.37}
$$

then $t_{rm}(t_o)$ has exactly one local minimum in $(0, \pi/\Omega)$, say at $t_o = t_{o,min}(T)$. Now since the expression for the derivative in the first line of Eq. (7.36) is zero iff t_o is equal to

$$
t_{o,min}^{[1]}(T) = (\pi/\Omega)[T/(\pi/\Omega)]^{1/3}, \tag{7.38}
$$

and $t_{o,min}^{[1]}(T) \in [0, (1/2)(\pi/\Omega)]$ iff $T/(\pi/\Omega) \le 1/8$, then $t_{o,min}(T) =$ is given by $t_{o,min}^{[1]}(T)$ whenever $T/(\pi/\Omega) \le 1/8$:

$$
t_{o,min}(T) = t_{o,min}^{[1]}(T) = (\pi/\Omega)[T(\pi/\Omega)]^{1/3} \text{ if } T/(\pi/\Omega) \le 1/8. \tag{7.39}
$$

On the other hand, if $T/(\pi/\Omega) > 1/8$ then $(dt_{rm}/dt_o)((1/2)(\pi/\Omega)) < 0$ so, using Eq. (7.37), we conclude that

$$t_{o,min}(T) \in ((1/2)(\pi/\Omega), \pi/\Omega) \tag{7.40}$$

and is given by the appropriate root, $t^{[2]}_{o,min}(T)$, of the quartic equation

$$4[T/(\pi/\Omega)]\{1 - [t_o/(\pi/\Omega)]\} = \left(1 - 2\{1 - [t_o/(\pi/\Omega)]\}^2\right)^2, \tag{7.41}$$

namely,

$$t_{o,min}(t) = t^{[2]}_{o,min}(T) \text{ if } T/(\pi/\Omega) \geq 1/8. \tag{7.42}$$

We will display $t^{[2]}_{o,min}(T)$ shortly, but we now wish to make an important point: for our triangular pulse, Eq. (7.40) tells us that $\bar{t}_o < \pi/\Omega$, always; i.e., *in our model with cutoff we may never encompass the full triangular pulse* but only some proper initial portion of it. On the other hand, we see from Eq. (7.41) (or from Eqs. (7.46) - (7.48) below) that

$$\lim_{T/(\pi/\Omega)\to\infty} t^{[2]}_{o,min}(T) = \pi/\Omega \tag{7.43}$$

so we can in principle get as close as we please to the full pulse by taking $T/(\pi/\Omega)$ $[= 2\kappa^{-1}(v_o/c)/(\pi/\Omega)^2 AY]$ large enough. In analogy with the constant pulse, we call the case $T \leq (1/8)(\pi/\Omega)$ the "superstrong" (SS) case (because here the cutoff occurs during the rising portion of the pulse) and the case $T > (1/8)(\pi/\Omega)$ the strong (S) case (here the cutoff occurs during the falling portion of the pulse); there is no weak case but we also distinguish the case of the "weak limit" (WL), where $T/(\pi/\Omega) \to \infty$. We then have, in analogy with Eqs. (7.7) - (7.9),

$$\bar{t}_o = \begin{cases} t^{[1]}_{o,\min}(T), & \text{if } T \le (1/8)(\pi/\Omega) \quad \text{(SS)} \\ t^{[2]}_{o,\min}(T), & \text{if } T > (1/8)(\pi/\Omega) \quad \text{(S)} \\ \pi/\Omega, & \text{if } T/(\pi/\Omega) \to \infty \quad \text{(WL)} \end{cases} \tag{7.44}$$

and

$$t_{rm}(\bar{t}_o) = \begin{cases} (3/2)t^{[1]}_{o,\min}(T), & \text{if SS} \\ t^{[2]}_{o,\min}(T) + T\{1 - 2[1 - t^{[2]}_{o,\min}(T)/(\pi/\Omega)]^2\}^{-1}, & \text{if S} \\ (\pi/\Omega) + T, & \text{if WL} \end{cases} \tag{7.45}$$

and we may proceed by cases to the radiation quantities. Finally, as promised, we have

$$t^{[2]}_{o,\min}(T)/(\pi/\Omega) = 1 - \tfrac{1}{2}\left(R - \{2 - R^2 + 2[T/(\pi/\Omega)]R^{-1}\}^{1/2}\right) \tag{7.46}$$

where

$$R = +[\tfrac{2}{3} + 2^{-1/3}(\hat{R}_+ + \hat{R}_-)]^{1/2} \tag{7.47}$$

for

$$\hat{R}_\pm = \left(\{\tfrac{16}{27} + [T/(\pi/\Omega)]^2\} \pm [T/(\pi/\Omega)]\{\tfrac{32}{27} + [T/(\pi/\Omega)]^2\}^{1/2}\right)^{1/3}. \tag{7.48}$$

D. Parabolic

The parabolic pulse

$$\hat{f}(t) = 4A[t/(\pi/\Omega)]\{1 - [t/(\pi/\Omega)]\}, \quad t \in [0, \pi/\Omega], \quad A > 0, \tag{7.49}$$

yields results very similar to those for the triangular pulse. Once again no weak pulse is possible; further, since the parabolic pulse is defined as one piece, it is not as natural here to distinguish between a superstrong and strong pulse (although this of course may be done) as in the triangular case; we may thus consider only a strong pulse and the weak limit. Also, we have

$$(dt_{ret}/dt_o)(t_o) = 1 - 6[T/(\pi/\Omega)]\{1 - [t_o/(\pi/\Omega)]\} \; / \; [t_o/(\pi/\Omega)]^3\Big(1 + 2\{1 - [t_o/(\pi/\Omega)]\}\Big)^2 \tag{7.50}$$

with

$$T = 3/(\pi/\Omega)\omega_p^2 \tag{7.51}$$

and $(d^2t_{ret}/dt^2)(t_o) > 0$ on $[0, \pi/\Omega]$ so determination of unique $t_{o,min}(T)$ involves solving the quintic in $t_o/(\pi/\Omega)$ derived from Eq. (7.50) and this cannot be done analytically in general -- it must be done numerically.

E. Sine-squared

The pulse

$$\hat{I}(t) = A\sin^2\{\pi[t/(\pi/\Omega)]\} = A\sin^2(\Omega t), \quad t \in [0, \pi/\Omega], \qquad A > 0, \tag{7.52}$$

yields results similar to those for the parabolic pulse, exhibiting only a strong pulse and a weak limit. To find $t_{o,min}(T)$ we must solve

$$\sin\Omega t_o/[2\Omega t_o - \sin 2\Omega t_o] = 1/2\sqrt{2}\,\pi[T/(\pi/\Omega)]^{1/2} \tag{7.53}$$

($x - \sin x = 0$ at $x = 0$ and $(d/dx)(x - \sin x) = 1 - \cos x > 0$ on $(0, \pi)$ so $x - \sin x > 0$ there); this has unique solution since $\sin \Omega t_o/[2\Omega t_o - \sin 2\Omega t_o]$ strictly decreases from ∞ to 0 on $[0, \pi/\Omega] \ni t_o$, but of course it must be found numerically in general.

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APPENDIX A: A SPECIAL CLASS OF PULSES

In this appendix we exhibit a class of pulses each member of which has the property that its associated map $t_o \mapsto t_{rm}(t_o)$, $t_o \in [0, \bar{t}_o]$, possesses a positive-length interval of constancy and, furthermore, $\bar{t}_o = \pi/\Omega$ (the full pulse width). Classes with these properties other than the one we will exhibit also exist but we will present only one such class. In particular, we will prove the following.

Theorem A.1: Fix Ω, v_o, $Y > 0$.

(i) If $T \geq \pi/\Omega$, $0 < \alpha_1 < 2/3$, and $\alpha_1 < \alpha_2 < \pi/\Omega T$ then the pulse

$$I_{\alpha_1\alpha_2}(s) = 2\Omega^2\{1/[e^2(Y/v_o)/m_e\varepsilon_o]\}p_{\alpha_1\alpha_2}(s)\ , \qquad s \in [0, \pi], \tag{A.1}$$

where

$$p_{\alpha_1\alpha_2}(s) = \begin{cases} [1/(\Omega T)^2(1-\alpha_1)^2\alpha_1]\{-2[(1-2\alpha_1)/\alpha_1\Omega T]s + [2-3\alpha_1]\}, & \text{if } s \in [0, \alpha_1\Omega T] \\ 1/(\Omega T - s)^2\ , & \text{if } s \in [\alpha_1\Omega T, \alpha_2\Omega T] \\ [1/(\Omega T)^3(1-\alpha_2)^2\alpha_2]s\ , & \text{if } s \in [\alpha_2\Omega T, \pi], \end{cases} \tag{A.2}$$

has the property

$$t_{rm,\alpha_1\alpha_2} = T \text{ for all } t_o \in [\alpha_1 T,\ \alpha_2 T]. \tag{A.3}$$

(ii) If, in addition, $T \geq 3(\pi/\Omega)$ and $\alpha_1 < \pi/\Omega T$ ($\leq 1/3$) then there exists $\alpha_2^* \in (\alpha_1, \pi/\Omega T)$ such that if $\alpha_2 \in (\alpha_2^*, \pi/\Omega T)$ then (a) $\bar{t}_o = \bar{t}_o[T, G_{\alpha_1\alpha_2}] = \pi/\Omega$, where $T = 2/\omega_p^2 N_{\alpha_1\alpha_2}(\Omega)$ (see Eqs. (3.25) and (3.36)); and (b) $t_{rm,\alpha_1\alpha_2}(\bar{t}_o) < T$.

Proof: (i) It is easy to check that $p_{\alpha_1\alpha_2}$ is continuous on $[0, \pi]$ and also (strictly) positive

there (since $\alpha_1 < 2/3$). Next, denoting

$$A = \max_{s \in [0,\pi]} I_{\alpha_1\alpha_2}(s) \tag{A.4}$$

we rewrite $I_{\alpha_1\alpha_2}$ as

$$I_{\alpha_1\alpha_2}(s) = A\{2\Omega^2/[e^2(AY/v_o)/m_e\varepsilon_o]\}p_{\alpha_1\alpha_2}(s) \equiv Ag_{\alpha_1\alpha_2}(s) \tag{A.5}$$

and, noting that

$$\max_{s \in [0,\pi]} g_{\alpha_1\alpha_2}(s) = 1, \tag{A.6}$$

we then have

$$g_{\alpha_1\alpha_2}(s) = 2(\Omega^2/\omega_p^2)p_{\alpha_1\alpha_2}(s). \tag{A.7}$$

Noticing from Eq. (3.25) that

$$2(\Omega^2/\omega_p^2) = \Omega N_{\alpha_1\alpha_2}(\Omega) \cdot \Omega T \tag{A.8}$$

and from Eqs. (2.4) and (2.6) that

$$\Omega N_{\alpha_1\alpha_2}(\Omega)G_{\alpha_1\alpha_2}(s) = \int_0^s g_{\alpha_1\alpha_2}(s')ds', \qquad s \in [0,\pi], \tag{A.9}$$

we find from Eqs. (A.1) and (A.7) that

$$
G_{\alpha_1\alpha_2}(s)/\Omega T = \begin{cases} [1/(\Omega T)^2(1-\alpha_1)^2\alpha_1]\{-[(1-2\alpha_1)/\alpha_1\Omega T]s^2 + [2-3\alpha_1]s\}, & \text{if } s \in [0, \alpha_1 \Omega T] \\ 1/(\Omega T - s), & \text{if } s \in [\alpha_1\Omega T, \alpha_2\Omega T] \\ (1/2)[1/\Omega T(1-\alpha_2)^2]\{[1/\alpha_2(\Omega T)^2]s^2 + [2-3\alpha_2]\}, & \text{if } s \in [\alpha_2\Omega T, \pi]. \end{cases}
$$

(A.10)

Hence, using

$$
\Omega t_{rm}(t_o) = \Omega t_o + [\Omega T/G(\Omega t_o)], \tag{A.11}
$$

we find

$$
\Omega t_{rm,\alpha_1\alpha_2}(t_o) = \begin{cases} \Omega t_o + (\Omega T)^2(1-\alpha_1)^2\alpha_1/\{-[(1-2\alpha_1)/\alpha_1\Omega T](\Omega t_o)^2 + [2-3\alpha_1](\Omega t_o)\}, & \text{if } t_o \in [0, \alpha_1 T] \\ \Omega T, & \text{if } t_o \in [\alpha_1 T, \alpha_2 T] \\ \Omega t_o + 2(\Omega T)(1-\alpha_2)^2/\{[1/\alpha_2(\Omega T)^2](\Omega t_o)^2 + [2-3\alpha_2]\}, & \text{if } t_o \in [\alpha_2 T, \pi/\Omega] \end{cases}
$$

(A.12)

so Eq. (A.3) holds. (Note that $t_{rm,\alpha_1\alpha_2}(0) = \infty$, as required, but that the denominators in the first and third lines of Eq. (A.12) are never 0 since $\alpha_1, \alpha_2 \in (0, 2/3)$.)

(ii) Let $T \geq 3(\pi/\Omega)$. To prove (a), we will show that if $0 < \alpha_1 < \pi/\Omega T$ then there exists $\alpha_2^* \in (\alpha_1, \pi/\Omega T)$ such that if $\alpha_2 \in (\alpha_2^*, \pi/\Omega T)$ then $(dt_{rm,\alpha_1\alpha_2}/dt_o)(t_o) \leq 0$ for all $t_o \in (0, \pi/\Omega]$ so that $\bar{t}_o = \pi/\Omega$, by Eq. (3.36); note that for such α_1, α_2 the conditions of (i) hold so that $t_{rm,\alpha_1\alpha_2}$ enjoys the property given by Eq. (A.3).

We begin by noting that, for $0 < \alpha_1 < 2/3$ and $\alpha_1 < \alpha_2 < \pi/\Omega T$, $t_{rm,\alpha_1\alpha_2}(t_o)$ is differentiable on $(0, \pi/\Omega]$ since $g_{\alpha_1\alpha_2}$ is continuous there. The stationary points of $t_{rm,\alpha_1\alpha_2}$ are thus found from

$$(dt_{m,\alpha_1\alpha_2}/dt_o)(t_o) = 1 - \Omega T G'_{\alpha_1\alpha_2}(s)/G^2_{\alpha_1\alpha_2}(s) = 0, \quad \Omega t_o = s \in (0,\pi), \tag{A.13}$$

where, from Eq. (A.10),

$$\Omega T G'_{\alpha_1\alpha_2}(s)/G^2_{\alpha_1\alpha_2}(s) = \begin{cases} (\Omega T)^2(1-\alpha_1)^2\alpha_1\{-2[(1-2\alpha_1)/\alpha_1\Omega T]s + [2-3\alpha_1]\} \\ \div s^2\{-[(1-2\alpha_1)/\alpha_1\Omega t]s + [2-3\alpha_1]\}^2, & \text{if } s \in (0, \alpha_1\Omega T] \\ 1, & \text{if } s \in [\alpha_1\Omega T, \alpha_2\Omega T] \\ 4[(1-\alpha_2)^2/\alpha_2\Omega t]s/\{[1/\alpha_2(\Omega T)^2]s^2 + [2-3\alpha_2]\}^2, & \text{if } s \in [\alpha_2\Omega T, \pi]. \end{cases} \tag{A.14}$$

If $s \in (0, \alpha_1 \Omega T]$ then Eq. (A.13) becomes, using

$$\sigma \equiv s/\alpha_1\Omega T \qquad (0 \le \sigma \le 1), \tag{A.15}$$

$$\mathcal{S}_4(\sigma) \equiv (1-2\alpha_1)^2\sigma^4 - 2(1-2\alpha_1)(2-3\alpha_1)\sigma^3 + (2-3\alpha_1)^2\sigma^2 + (2/\alpha_1)(1-\alpha_1)^2(1-2\alpha_1)\sigma$$
$$- (1/\alpha_1)(1-\alpha_1)^2(2-3\alpha_1) = 0 = (\sigma-1)\mathcal{S}_3(\sigma) \tag{A.16}$$

for

$$\mathcal{S}_3(\sigma) \equiv (1-2\alpha_1)^2\sigma^3 - (1-2\alpha_1)(3-4\alpha_1)\sigma^2 + (1-\alpha_1)^2\sigma + (1/\alpha_1)(1-\alpha_1)^2(2-3\alpha_1). \tag{A.17}$$

Hence $\sigma = 1$, i.e., $t_o = \alpha_1 T$, gives a stationary point of $t_{m,\alpha_1\alpha_2}(t_o)$. Further, as we demonstrate in Lemma A.2 below, $\mathcal{S}_3(\sigma)$ has no roots in [0, 1] when $0 < \alpha_1 < 1/3$ so that $\sigma = 1$ is the only root of $\mathcal{S}_4(\sigma)$ in [0, 1]; i.e., $t_o = \alpha_1 T$ is the only stationary point of $t_{m,\alpha_1\alpha_2}$ in $[0, \pi/\Omega]$. Since $dt_{m,\alpha_1\alpha_2}/dt_o$ is

continuous on $(0, \alpha_1 T]$ and, from Eqs. (A.13) and (A.14), $\lim_{t_o \to 0^+}(dt_{rm,\alpha_1\alpha_2}/dt_o)(t_o) = -\infty$ then we have

$$(dt_{rm,\alpha_1\alpha_2}/dt_o)(t_o) < 0 \text{ for } t_o \in (0, \alpha_1 T) \qquad (0 < \alpha_1 < 1/3). \tag{A.18}$$

Next, if $s \in [\alpha_1 \Omega T, \alpha_2 \Omega T]$ then Eq. (A.13) becomes $1 = 1$, which is satisfied for all $s \in [\alpha_1 \Omega T, \alpha_2 \Omega T]$, i.e., for all $t_o \in [\alpha_1 T, \alpha_2 T]$; of course this also follows directly from Eq. (A.12). So we have

$$(dt_{rm,\alpha_1\alpha_2}/dt_o)(t_o) = 0 \quad \text{for } t_o \in [\alpha_1 T, \alpha_2 T]. \tag{A.19}$$

Lastly, if $s \in [\alpha_2 \Omega T, \pi]$ then, using

$$\hat{\sigma} \equiv s/\alpha_2 \Omega T \qquad (1 \le \hat{\sigma} \le \pi/\alpha_2 \Omega T), \tag{A.20}$$

Eq. (A.13) becomes

$$\mathcal{H}_4(\hat{\sigma}) \equiv \alpha_2^2 \hat{\sigma}^4 + 2\alpha_2(2 - 3\alpha_2)\hat{\sigma}^2 - 4(1 - \alpha_2)^2 \hat{\sigma} + (2 - 3\alpha_2)^2 = 0 = (\hat{\sigma} - 1)\mathcal{H}_3(\hat{\sigma}) \tag{A.21}$$

for

$$\mathcal{H}_3(\hat{\sigma}) \equiv \alpha_2^2 \hat{\sigma}^3 + \alpha_2^2 \hat{\sigma}^2 + \alpha_2(4 - 5\alpha_2)\hat{\sigma} - (2 - 3\alpha_2)^2. \tag{A.22}$$

Hence $\hat{\sigma} = 1$, i.e., $t_o = \alpha_2 T$, gives a stationary point of $t_{rm,\alpha_1\alpha_2}(t_o)$, in agreement with Eq. (A.19). Further, since

$$(d\mathcal{H}_4/d\hat{\sigma})\,(1) = -12[(1/3) - \alpha_2)](1 - \alpha_2) \tag{A.23}$$

then $\hat{\sigma} = 1$ gives a local maximum iff $\alpha_2 \in (0, 1/3)$. We now rewrite $\mathcal{H}_5(\hat{\sigma})$, using

$$\Sigma \equiv \hat{\sigma} - 1 \qquad [0 \le \Sigma \le (\pi/\alpha_2 \Omega T) - 1], \tag{A.24}$$

to get

$$\begin{aligned}(1/\alpha_2^2)\mathcal{H}_5(\hat{\sigma}) &= (\hat{\sigma} - 1)^3 + 4(\hat{\sigma} - 1)^2 + (4/\alpha_2)(\hat{\sigma} - 1) - 4[(1/\alpha_2) - 1][(1/\alpha_2) - 3] \\ &= \Sigma^3 + 4\Sigma^2 + (4/\alpha_2)\Sigma - 4[(1/\alpha_2) - 1][(1/\alpha_2) - 3] \equiv \mathcal{H}_{51}(\Sigma)\end{aligned} \tag{A.25}$$

and seek the zeros of $\mathcal{H}_{51}$. Using standard techniques for cubic equations we find for Eq. (A.25) the discriminant

$$\Delta(\alpha_2) = (4/3)^3[(1/\alpha_2) - (4/3)]^3 + 4\{[(1/\alpha_2) - (4/3)]^2 + (1/27)\}^2 > 0 , \tag{A.26}$$

where the last inequality holds for $\alpha < 2/3$. Hence $\mathcal{H}_{51}$ has only one real zero in $(-\infty, \infty)$ and we denote the value of Σ that makes $\mathcal{H}_{51}(\Sigma) = 0$ by $\Sigma_o(\alpha_2)$. In fact,

$$\Sigma_o(\alpha_2) = \left(2\left\{[(\tfrac{1}{\alpha_2}) - (\tfrac{4}{3})]^2 + (\tfrac{1}{27})\right\} + \Delta^{1/2}(\alpha_2)\right)^{1/3} + \left(2\left\{[(\tfrac{1}{\alpha_2}) - (\tfrac{4}{3})]^2 + (\tfrac{1}{27})\right\} - \Delta^{1/2}(\alpha_2)\right)^{1/3} - (\tfrac{4}{3}) \tag{A.27}$$

but we do not need this explicit expression for our proof. (However, it is of practical use -- see later.) We will demonstrate in Lemma A.3 below that

$$\Sigma_o(\alpha_2) > 0 \text{ for } 0 < \alpha_2 \le \pi/\Omega T; \tag{A.28}$$

we then have

$$\alpha_2 + \alpha_2\Sigma_o(\alpha_2) > \alpha_2 \text{ for } 0 < \alpha_2 \le \pi/\Omega T \tag{A.29}$$

so that, in particular,

$$\pi/\Omega T + (\pi/\Omega T)\Sigma_o(\pi/\Omega T) > \pi/\Omega T. \tag{A.30}$$

Since $\alpha_2 \mapsto \alpha_2 + \alpha_2\Sigma_o(\alpha_2)$ is continuous on $[0, \pi/\Omega T]$ then there exists $\alpha_2^* \in (\alpha_1, \pi/\Omega T)$ such that

$$\alpha_2 + \alpha_2\Sigma_o(\alpha_2) > \pi/\Omega T \quad \text{for } \alpha_2 \in (\alpha_2^*, \pi/\Omega T)\ , \tag{A.31}$$

so that

$$\Sigma_o(\alpha_2) > \pi/\alpha_2\Omega T - 1 \quad \text{for } \alpha_2 \in (\alpha_2^*, \pi/\Omega T); \tag{A.32}$$

hence

$$\hat{\sigma}_o(\alpha_2) = \Sigma_o(\alpha_2) + 1 > \pi/\alpha_2\Omega T \quad \text{for } \alpha_2 \in (\alpha_2^*, \pi/\Omega T) \tag{A.33}$$

and so $\hat{\sigma} = 1$ is the only root of $\mathcal{H}_4$ in $[1, \pi/\alpha_2\Omega T]$. Thus $t_o = \alpha_2 T$ is the only stationary point of $t_{m,\alpha_1\alpha_2}(t_o)$ in $[\alpha_2 T, \pi/\Omega]$. Further, since $\hat{\sigma} = 1$ gives a local max for $\mathcal{H}_4$ (when $\alpha_2 < 1/3$) and $\mathcal{H}_4(1) = 0$, then

$$\mathcal{H}_4(\hat{\sigma}) < 0 \text{ for } \hat{\sigma} \in (1, \pi/\alpha_2\Omega T] \tag{A.34}$$

since $\mathcal{H}_4$ is continuous on $[1, \pi/\alpha_2\Omega T]$. But

$$(dt_{m,\alpha_1\alpha_2}/dt_o)(t_o) = \mathcal{H}_4(t_o/\alpha_2 T)/[\alpha_2(t_o/\alpha_2 T)^2 - (3\alpha_2 - 2)]^2, \quad t_o \in [\alpha_2 T, \pi/\Omega] \tag{A.35}$$

so finally we have, using Eq. (A.34),

$$(dt_{m,\alpha_1\alpha_2}/dt_o)(t_o)(t_o) < 0 \quad \text{for } t_o \in (\alpha_2 T, \pi/\Omega] \quad (\alpha_2^* < \alpha_2 < \pi/\Omega T). \tag{A.36}$$

Taken together, Eqs. (A.18), (A.19), and (A.36) show $\bar{t}_o = \pi/\Omega$, so (a) is proved.

To show (b), we first note that since, by the above, $\bar{t}_o = \pi/\Omega$, and since

$$t_{m,\alpha_1\alpha_2}(\pi/\Omega) = \pi/\Omega + T = \pi/\Omega + 2/\omega_p^2 N_{\alpha_1\alpha_2}(\Omega) \tag{A.37}$$

then we must show that

$$\pi/\Omega + 2/\omega_p^2 N_{\alpha_1\alpha_2}(\Omega) < T. \tag{A.38}$$

Computing

$$N_{\alpha_1\alpha_2}(\Omega) = (1/\Omega)\int_0^{\pi} g_{\alpha_1\alpha_2}(s)ds = [1/\omega_p^2 T(1-\alpha_2)^2][2 - 3\alpha_2 + (1/\alpha_2)(\pi/\Omega T)^2], \tag{A.39}$$

Eq. (A.37) becomes, for $\alpha_2 < 2/3$,

$$-\alpha_2^3 + (1/2)[1 + 3(\pi/\Omega T)]\alpha_2^2 - (\pi/\Omega T)\alpha_2 + (1/2)(\pi/\Omega T)^2[1 - (\pi/\Omega T)] > 0. \tag{A.40}$$

We will demonstrate below in Lemma A.4 that this last equation indeed holds if $\pi/\Omega T \le 1/3$ and $\alpha_2 < \pi/\Omega T$ so that (b) is proved. ■

We now present the three lemmas referred to in the proof of the theorem.

Lemma A.2: Let $0 < \alpha < 1/3$ and for $\sigma \in \mathbb{R}$ define

$$F_\alpha(\sigma) \equiv (1-2\alpha)^2\sigma^3 - (1-2\alpha)(3-4\alpha)\sigma^2 + (1-\alpha)^2\sigma + (1/\alpha)(1-\alpha)^2(2-3\alpha). \tag{A.41}$$

Then $F_\alpha(\sigma) > 0$ for $\sigma \in [0, 1]$.

Proof: From $(1/3)\alpha(4-5\alpha) > 0$ it follows that

$$\begin{aligned}(1-\alpha)^2 &= 1-2\alpha+\alpha^2 > 1-(10/3)\alpha + (8/3)\alpha^2 = (8/3)[(3/8)-(5/4)\alpha+\alpha^2]\\ &= (8/3)[(1/2)-\alpha][(3/4)-\alpha];\end{aligned} \tag{A.42}$$

noting $(2/3-\alpha) > \alpha$ we then have

$$(2-3\alpha)(1-\alpha)^2 > \alpha(1-2\alpha)(3-4\alpha). \tag{A.43}$$

Hence, for $\sigma \in [\,0, 1]$,

$$(1/\alpha)(2-3\alpha)(1-\alpha)^2 > (1-2\alpha)(3-4\alpha) \ge (1-2\alpha)(3-4\alpha)\sigma \tag{A.44}$$

so

$$\begin{aligned}&(1-2\alpha)^2\sigma^3 + (1-\alpha)^2\sigma + (1/\alpha)(1-\alpha)^2(2-3\alpha) \ge (1/\alpha)(1-\alpha)^2(2-3\alpha)\\ &> (1-2\alpha)(3-4\alpha)\sigma \ge (1-2\alpha)(3-4\alpha)\sigma^2\,,\end{aligned} \tag{A.45}$$

from which it follows immediately that $F_\alpha(\sigma) > 0$. ■

Lemma A.3: Let $0 < \alpha < 1/3$ and for $\Sigma \in \mathbb{R}$ define

$$H_\alpha(\Sigma) \equiv \Sigma^3 + 4\Sigma^2 + (4/\alpha)\Sigma - 4[(1/\alpha) - 1][(1/\alpha) - 3]. \tag{A.46}$$

Let $\Sigma_o(\alpha)$ denote the unique zero of H_α (as per Eq. (A.26) ff.). Then $\Sigma_o(\alpha) > 0$.

Proof: It is sufficient to show that $H_\alpha(\Sigma) < 0$ for all $\Sigma \leq 0$. To this end, we let $\overline{\Sigma} \equiv -\Sigma$ and

$$\overline{H}_\alpha\overline{\Sigma} \equiv -\alpha^2 H_\alpha(\overline{\Sigma}) = \alpha^2\overline{\Sigma}^3 - 4\alpha^2\overline{\Sigma}^2 + 4\alpha\overline{\Sigma} + 4(1-\alpha)(1-3\alpha),\ \ \overline{\Sigma} \in \mathbb{R} \tag{A.47}$$

and show

$$\overline{H}_\alpha(\overline{\Sigma}) > 0 \text{ for all } \overline{\Sigma} \geq 0. \tag{A.48}$$

If $\overline{\Sigma} = 0$ then Eq. (A.48) clearly holds, so suppose $\overline{\Sigma} > 0$. Now it is clear that

$$\alpha\overline{\Sigma}^2 - 4\alpha\overline{\Sigma} + 4 > 0 \text{ for all } \overline{\Sigma} \in \mathbb{R} \tag{A.49}$$

(considered as a parabola in $\overline{\Sigma}$) so that

$$\alpha^2\overline{\Sigma}^3 + 4\alpha\overline{\Sigma} > 4\alpha^2\overline{\Sigma}^2 \text{ for } \overline{\Sigma} > 0; \tag{A.50}$$

hence

$$\alpha^2\overline{\Sigma}^3 + 4\alpha\overline{\Sigma} + 4(1-\alpha)(1-3\alpha) - 4\alpha^2\overline{\Sigma}^2 > 0 \quad \text{for } \overline{\Sigma} > 0 \tag{A.51}$$

and this is precisely Eq. (A. 48). ■

Lemma A.4: Let $\pi/\Omega T \leq 1/3$ and for $\alpha \in \mathbb{R}$ define

$$J_{\pi/\Omega T}(\alpha) \equiv -\alpha^3 + (1/2)[1 + 3(\pi/\Omega T)]\alpha^2 - (\pi/\Omega T)\alpha + (1/2)(\pi/\Omega T)^2[1 - (\pi/\Omega T)]. \tag{A.52}$$

Then $J_{\pi/\Omega T}(\alpha) > 0$ for $\alpha \in (0, \pi/\Omega T)$.

Proof: This follows immediately from

$$J_{\pi/\Omega T}(0) = (1/2)(\pi/\Omega T)^2[1 - (\pi/\Omega T)] > 0, \tag{A.53}$$

$$J_{\pi/\Omega T}(\pi/\Omega T) = 0, \tag{A.54}$$

and

$$(dJ_{\pi/\Omega T}/d\alpha)(\alpha) = -3[(1/3) - \alpha][\pi/\Omega T) - \alpha] \tag{A.55}$$

since the derivative is negative for $\alpha < \min\{1/3, \pi/\Omega T\}$. ■

As a concrete example of a pulse in the class of the theorem, we choose $T = 4(\pi/\Omega)$, $\alpha_1 = 1/32$, and $\alpha_2 = 1/8$; that such an α_2 is adequate can be verified by the fact that it satisfies Eq. (A.31), with $\Sigma_o(\alpha_2)$ given by Eq. (A.27). We leave Ω, Y, v_o, hence *A* and ω_p^2, nonspecific. We then find

$$g_{1/32,\,1/8}(s) = 2(\Omega/\omega_p)^2(1/\pi^2) \times \begin{cases} (8/31)^2[-480(s/\pi) + 61], & \text{if } s \in [0, \pi/8] \\ 1/[4 - (s/\pi)]^2, & \text{if } s \in [\pi/8, \pi/2] \\ (8/49)(s/\pi), & \text{if } s \in [\pi/2, \pi], \end{cases} \tag{A.56}$$

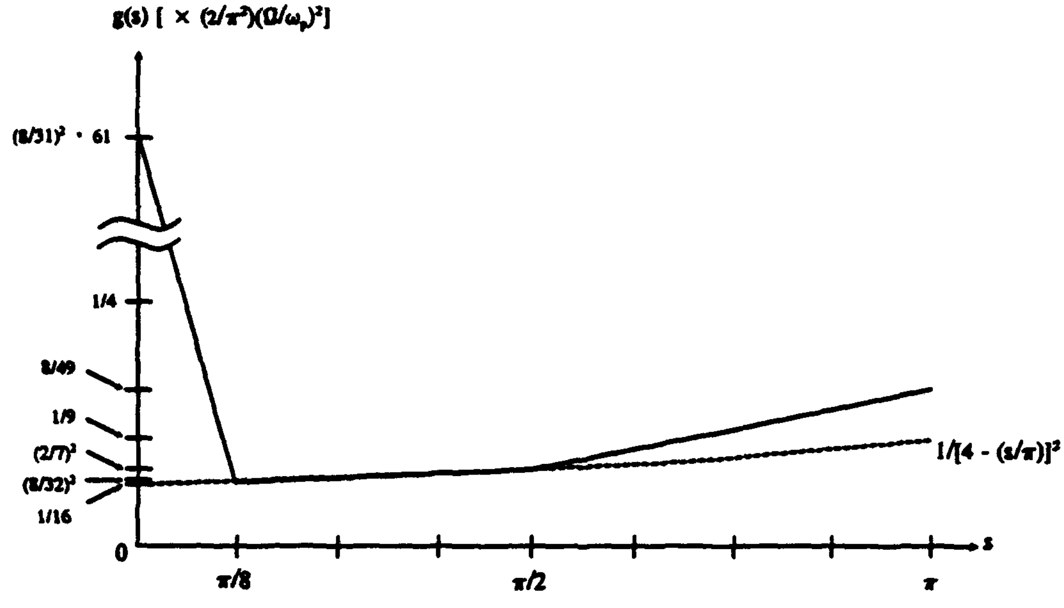
$$\Omega N_{1/32,\,1/8}(\Omega) = (34/49\pi)(\Omega/\omega_p)^2, \tag{A.57}$$

$$G_{1/32,\,1/8}(s) = (49/17) \times \begin{cases} (8/31)^2[-240(s/\pi)^2 + 61(s/\pi)], & \text{if } s \in [0, \pi/8] \\ 1/[4 - (s/\pi)], & \text{if } s \in [\pi/8, \pi/2] \\ (1/49)[4(s/\pi)^2 + 13], & \text{if } s \in [\pi/2, \pi], \end{cases} \tag{A.58}$$

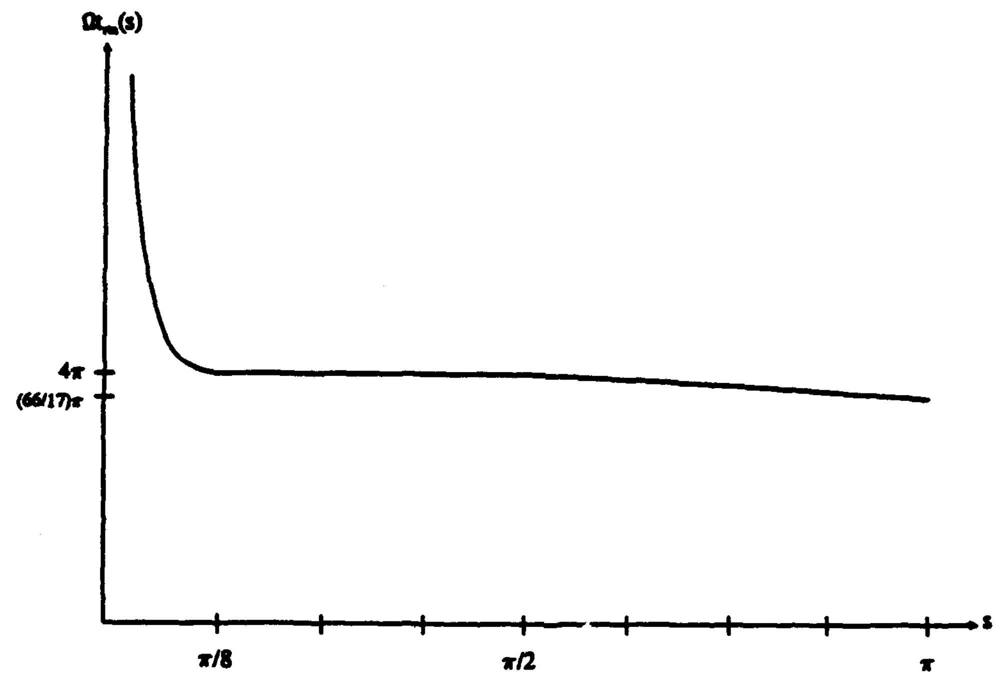
and

$$\Omega t_{rm,\,1/32,1/8}(s) = \begin{cases} \pi\{(s/\pi) + (31/8)^2[-240(s/\pi)^2 + 61(s/\pi)]^{-1}\}, & \text{if } s \in [0, \pi/8] \\ 4\pi, & \text{if } s \in [\pi/8, \pi/2] \\ \pi\{(s/\pi) + 49[4(s/\pi)^2 + 13]^{-1}\}, & \text{if } s \in [\pi/2, \pi]. \end{cases} \tag{A.59}$$

We illustrate g and Ωt_{rm} in Fig. 10.



(a) $g_{1/32,\,1/8}$



(b) $\Omega t_{rm,1/32,1/8}$

Figure 10. The pulse $I_{1/32,\,1/8}$ of Appendix A. Notice that g is not smooth at $s = \pi/2$, but Ωt_{rm} is smooth everywhere in $[0, \pi]$.

APPENDIX B: MATHEMATICAL COMPLEMENTS TO SECTION III

Proof of Theorem 3.2. (i) First note from Eq. (3.37) that

$$(dt_{rm}/dt_o)(t_o) = 1 - [T/G(\Omega t_o)][\ell n\ G(\Omega t_o)]' ; \tag{B.1}$$

since $\lim_{t_o \to 0^+} [T/G(\Omega t_o)] = \infty = \lim_{t_o \to 0^+} [\ell n\ G(\Omega t_o)]'$ then

$$\lim_{t_o \to 0^+} (dt_{rm}/dt_o)(t_o) = -\infty . \tag{B.2}$$

Also, since G' and G are continuous on $(0, \pi/\Omega)$ and G never vanishes there then G'/G^2, hence dt_{rm}/dt_o, is also continuous there. Therefore, there exists $\delta > 0$ such that $(dt_{rm}/dt_o)(t_o) < 0$ for all $t_o \in (0, \delta]$. So δ is a member of the set on the RHS of Eq. (3.36) and $\bar{t}_o \geq \delta > 0$, as required.

(ii) Since $\bar{t}_o > 0$ then $[0, \bar{t}_o] \neq \emptyset$. Let $t_o \in [0, \bar{t}_o]$; if $t_o = 0$ then t_o is admissible, so suppose $t_o > 0$. If $t_o' < t_o$ then from Fig. 2 we see that $\emptyset \neq \Gamma^o(t_o) \subseteq \Gamma^o(t_o')$, so suppose $t \in \Gamma^o(t_o)$ and note that $\partial Z/\partial t$ and $\partial^2 Z/\partial t^2$ exist at $(t_o'; t)$ and $(t_o; t)$. Now from Eq. (3.9) we have

$$(\partial/\partial t_o)(\partial^2 Z/\partial t^2)(t_o'; t) = -(e/m_e\varepsilon_o)\rho(Z(t_o'; t), t)(\partial Z/\partial t_o)(t_o'; t) \tag{B.3}$$

and from Eq. (3.21) and the fact that $g(s) = 0$ only for at most $s = 0, \pi$, it follows that, for $0 < t_o' < t_o$, LHS of Eq. (B.3) < 0; but $\rho(Z(t_o'; t), t) < 0$ (since charge sheet $\bar{t}_o$ is at $Z(t_o'; t)$ at time t). Hence

$$(\partial Z/\partial t_o)(t_o'; t) < 0. \tag{B.4}$$

In other words, $Z(t_o; t)$ is a strictly decreasing function of t_o at each $t_o' \in (0, t_o)$ so that for such t_o',

$$Z(t_o'; t) > Z(t_o; t) . \tag{B.5}$$

But also $Z(0;t) = v_o t > Z(t_o;t)$ so Eq. (B.5) holds as well for $t_o' = 0$. Hence t_o is admissible by Definition 3.1. Next, if $\bar{t}_o < \pi/\Omega$ then there exists $\varepsilon > 0$ such that if $t_o \in (\bar{t}_o, \bar{t}_o + \varepsilon)$ then $\bar{t}_o$ is not admissible; indeed, if $\varepsilon > 0$ is small enough (so that, at least, $\bar{t}_o + \varepsilon < \pi/\Omega$) it is clear from Fig. 2b (or from the definition of $\bar{t}_o$ in Eq. (3.36)) that, for $t_o \in (\bar{t}_o, \bar{t}_o+\varepsilon)$, sheet t_o leaves after and returns after sheet $\bar{t}_o$ so there exists $t \in I^o(t_o) \cap I^o(\bar{t}_o)$ such that $Z(t_o; t) = Z(\bar{t}_o; t)$; hence t_o is not admissible. Finally, if $\bar{t}_o = \pi/\Omega$ then $[0,\bar{t}_o] = [0, \pi/\Omega]$ which is clearly the largest possible interval of t_o's. ■

We have also shown (and will use later on):

Corollary B.1: If $t_o \in [0, \bar{t}_o]$ and $t \in I^o(t_o)$ then the map $t_o \mapsto Z(t_o; t)$ is strictly decreasing at t_o.

We next prove Theorem 3.3.

Proof of Theorem 3.3. If $(z, t) \in D_1[T, G]$, $z, t > 0$, the existence of a solution, say t_{o1}, in $[0, \bar{t}_o]$ follows immediately from the definitions of $D_1[T, G]$ and $D[T, G]$. If t_{o2} is another solution in $[0, \bar{t}_o]$ then $Z(t_{o1}; t) = z = Z(t_{o2}; t)$. But t_{o1} and t_{o2} are both admissible and if $z > 0$ then $t_{o1} \in I^o(t_{o1})$ and $t_{o2} \in I^o(t_{o2})$; hence $t_{o1} = t_{o2}$. Next, if $\rho(z, t) \neq 0$ then there exists $t_{oo} \in [0, \bar{t}_o]$ such that $Z(t_{oo}; t) = z$ so $(Z(t_{oo}; t), t) \in D_1[T, G]$, i.e., $(z, t) \in D_1[T, G]$; hence $(z, t) \notin D_1[T, G]$ implies $\rho(z, t) = 0$. Finally, if $z > 0$ and $t \leq 0$ then it is clear that $\rho(z, t) = 0$. ■

APPENDIX C: RETARDED TIME

In this Appendix, we establish the existence and uniqueness of solutions $\bar{t}'(t_o; \mathbf{x}, t)$ to Eq. (4.58) as well as derive smoothness properties of these solutions. Although our results may appear to be physically obvious, we eschew proof based on "obvious physical grounds". We fix

$$\mathbf{x} = (\xi, 0, \zeta), \quad \xi > 0, \quad \zeta \geq 0, \quad t \geq 0, \quad \text{and} \quad t_o \in [0, \bar{t}_o] \tag{C.1}$$

throughout unless otherwise noted. (We do not require $\xi > a$ in this Appendix C only.)

It is convenient to first consider the case $t_o = 0$. Eq. (4.58) then becomes

$$t - t' = \begin{cases} (1/c)[\xi^2 + (v_o t' - \zeta)^2]^{1/2}, & \text{if } t' \geq 0 \\ |\mathbf{x}|/c, & \text{if } t' < 0 \end{cases} \quad (t_o = 0) \tag{C.2}$$

and this has unique solution

$$\bar{t}'(0; \mathbf{x}, t) = \begin{cases} [1-(\frac{v_o}{c})^2]^{-1}\left(t-(\frac{v_o}{c})^2(\zeta/v_o)-\{[t-(\frac{v_o}{c})^2(\zeta/v_o)]^2 - [1-(\frac{v_o}{c})^2][t^2-(\frac{v_o}{c})^2(|\mathbf{x}|/v_o)^2]\}^{1/2}\right), & \text{if } t \geq |\mathbf{x}|/c \\ t - |\mathbf{x}|/c, & \text{if } t < |\mathbf{x}|/c \end{cases} \tag{C.3}$$

where the first line may be rewritten as

$$[1-(\tfrac{v_o}{c})^2]^{-1}\left(t-(\tfrac{v_o}{c})^2(\zeta/v_o) - (\tfrac{1}{c})\{[1-(\tfrac{v_o}{c})^2]\xi^2 + (v_o t-\zeta)^2\}^{1/2}\right) \qquad \text{if } t \geq |\mathbf{x}|/c; \tag{C.3a}$$

see Fig. 11.

We next consider the case $t_o \neq 0$. Rewriting Eq. (4.58) in detail we have

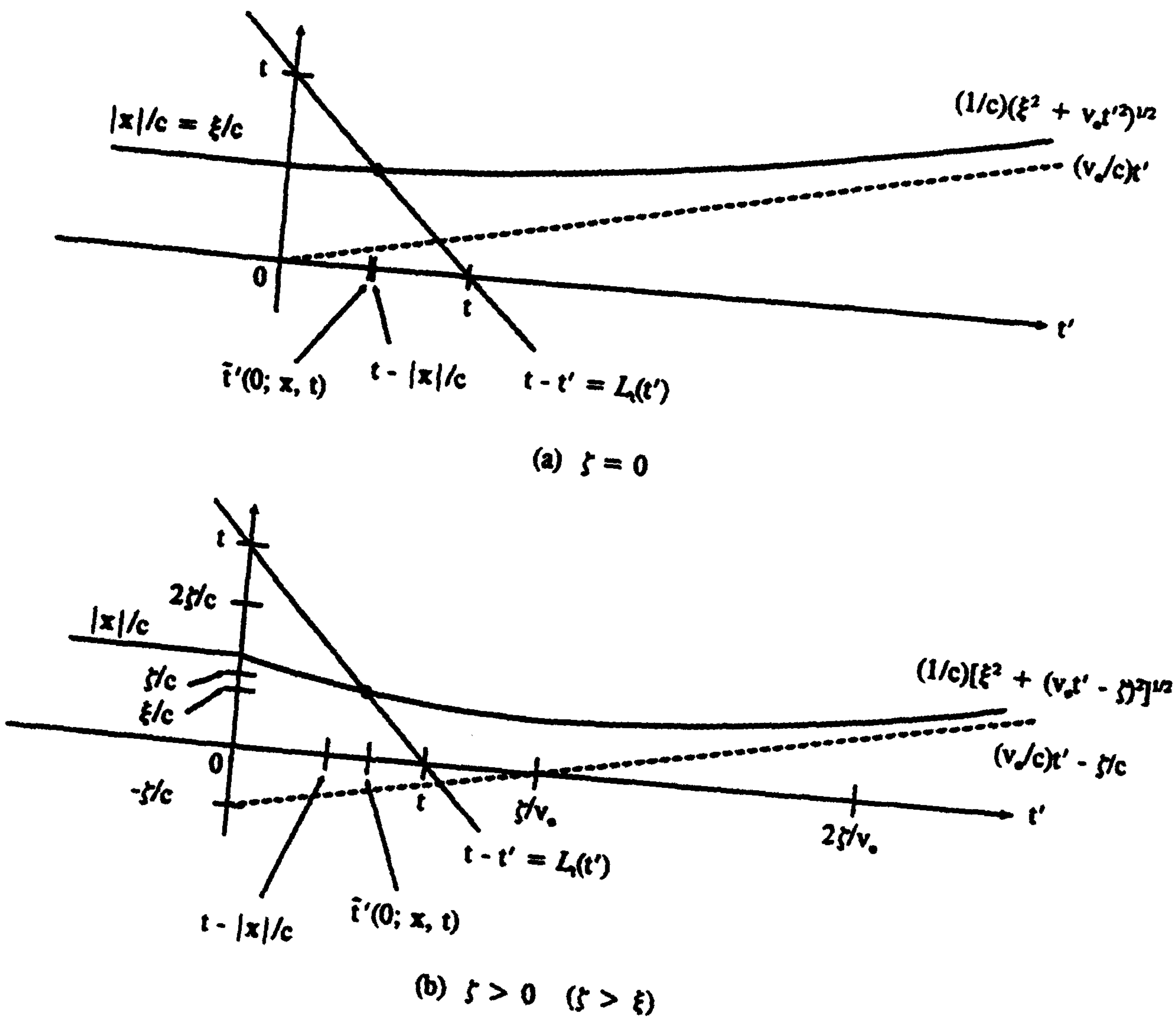


Figure 11. The unique solution to Eq. (C.2), given analytically by Eq. (C.3). For the case (b) we have arbitrarily taken, in addition, $\zeta > \xi$. The line $L_t(t')$ is drawn for the case $t - |x|/c > 0$.

$$
t - t' = \begin{cases} (1/c)\{\xi_s^2 + [v_o(t'-t_o) - (v_o/T)G(\Omega t_o)(t'-t_o)^2 - \zeta_s]^2\}^{1/2}, & \text{if } t' \in I(t_o) \\ |\mathbf{x}|/c, & \text{if } t' \in I^-(t_o) \end{cases} \quad (t_o \neq 0) \tag{C.4}
$$

where we have used Eq. (4.71). Unfortunately, in contrast to solving Eq. (C.2), which involves as an intermediary only a quadratic in t', solving Eq. (C.4) involves as an intermediary a quartic in t', the solution of which -- despite the availability of a general solution -- is extremely tedious to obtain; furthermore, we would still have to sort through these four solutions to determine which, if any, are solutions to Eq. (C.4). (These considerations are not unwarranted. For example, consider solving (*): $\alpha a - x = [a^2 - (x-a)^2]^{1/2}$, $a > 0$, $\alpha \in \mathbb{R}$. The associated quadratic has real solutions iff $\alpha \in [1-\sqrt{2}, 1+\sqrt{2}]$. Further, if $\alpha \in [2, 1+\sqrt{2}]$ then both of them solve (*); if $\alpha \in [0, 2)$ then only one of them solves (*); and if $\alpha \in [1-\sqrt{2}, 0)$ then neither solves (*).) Additionally, this $t_o \neq 0$ solution -- if Eq. (C.3) is any indicator -- is too complex to be integrated analytically in Eqs. (4.69) and (4.70). We will adopt the approach of not solving the intermediary quartic but rather of employing geometric reasoning (as in the solution of Eq. (C.2)) to establish the existence and uniqueness of solutions to Eq. (C.4); further, we will establish the smoothness properties that we require of the solution without ever actually obtaining it explicitly. In adopting this approach, we are abandoning the quest for expressions explicit in the inital pulse and surface parameters for the *finitely-remote* fields, $\mathbf{E}_s$ and $\mathbf{B}_s$. We are willing to do so because that will allow us to further progress towards our ultimate goal, namely, obtaining explicit expressions for the fields in the *radiation limit*; indeed, it will turn out that the explicit solution for $\bar{t}'$ when $t_o \neq 0$ is not needed to obtain the explicit expressions for the limit fields.

As a first step, we graph the RHS of Eq. (C.4); the curves of Fig. 12 include all generic possibilities. The curves are intentionally drawn to be "flat"; indeed, denoting the function on the RHS of Eq. (C.4) by $\mathcal{F}_{t_o}(t')$, we have, for $t' \in I^o(t_o)$,

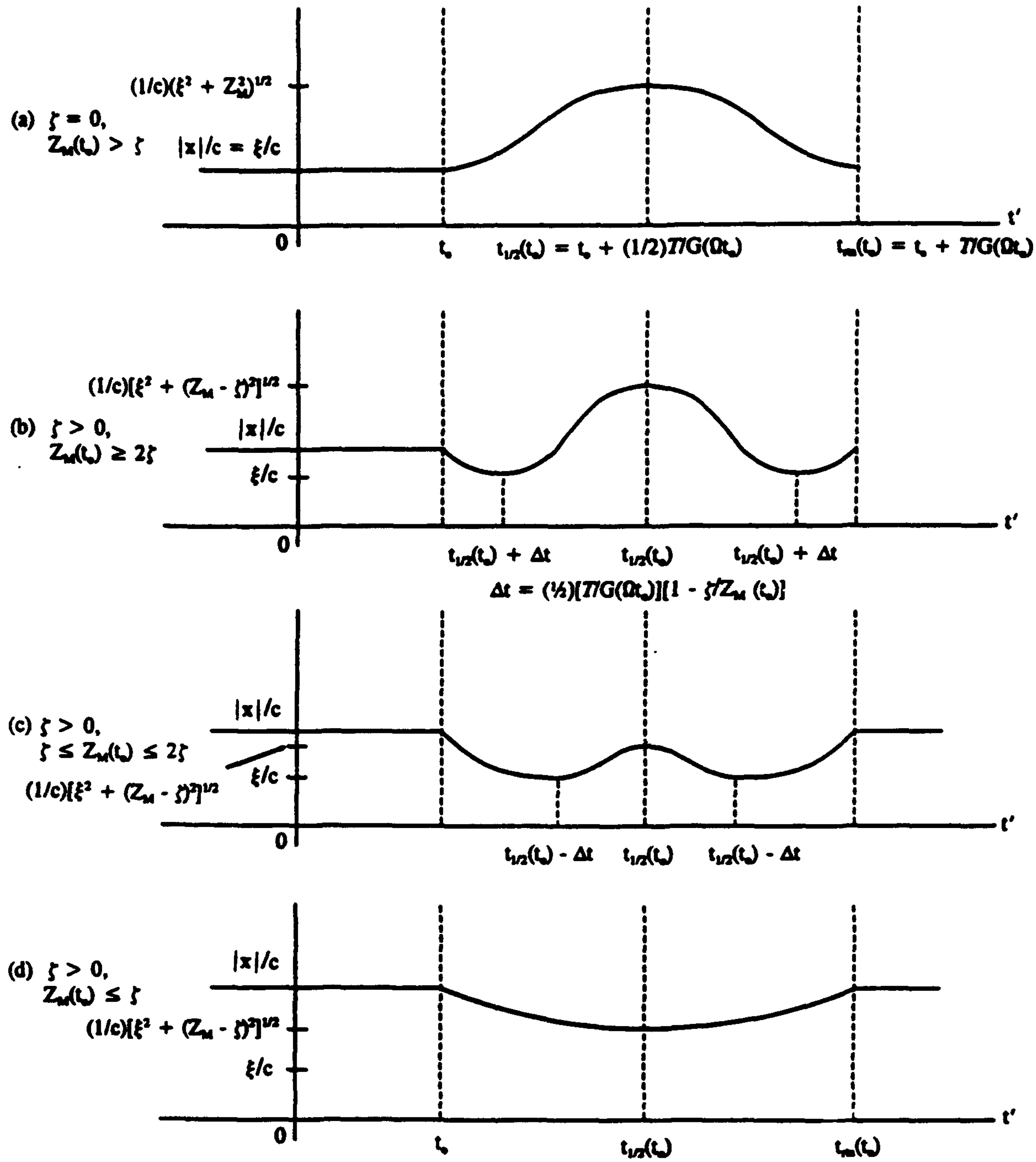


Figure 12. All generic possibilities for the RHS for Eq. (C.4). $Z_M(t_o)$ is the maximum z value attained by sheet t_o and is given by Eq. (3.22) as $Z_M(t_o) = (1/4)v_oT/G(\Omega t_o)$.

$$|d\mathcal{S}_{t_o}/dt'| = (\tfrac{v_o}{c})(1/c)|v_o(t'-t_o) - (v_o/T)G(\Omega t_o)(t'-t_o)^2 - \zeta|\cdot|1-2[G(\Omega t_o)/T](t'-t_o)|/\mathcal{S}_{t_o}(t')$$

$$< (\tfrac{v_o}{c})|1-2[G(\Omega t_o)/T](t'-t_o)| < \tfrac{v_o}{c} < 1 \tag{C.5}$$

since $|1-2[G(\Omega t_o)/T](t'-t_o)| < 1$ for $t'-t_o \in (0, T/G(\Omega t_o))$. The reader should note, however, that this figure, while enlightening, will not be the basis of our existence and uniqueness arguments; rather, they will be independent of this figure and based solely on Eq. (C.4).

We are now ready to prove the existence result.

Theorem C.1: Fix $\mathbf{x}$, t according to Eq. (C.1) and $t_o \in (0, \bar{t}_o]$. Then solutions to Eq. (C.4) exist.

Proof. First define, for $t' \leq t$,

$$L_t(t') = t - t' \tag{C.6}$$

and suppose first that $0 \leq t < |\mathbf{x}|/c$; then $(\mathcal{S}_{t_o} - L_t)(0) > 0$. On the other hand, $\lim_{t' \to -\infty} L_t(t') = \infty$ so there exists $t^* < 0$ such that $L_t(t^*) > |\mathbf{x}|/c = \mathcal{S}_{t_o}(t^*)$; hence $(\mathcal{S}_{t_o} - L_t)(t^*) < 0$. Since $\mathcal{S}_{t_o} - L_t$ is continuous on $(-\infty, \infty)$ then there exists $\bar{t} \in (t^*, 0)$ such that $(\mathcal{S}_{t_o} - L_t)(\bar{t}) = 0$, so that $\bar{t} < t$ solves Eq. (C.2). Similarly, if $t > |\mathbf{x}|/c$ then $(\mathcal{S}_{t_o} - L_t)(0) < 0$; and $L_t(t) = 0$, $\mathcal{S}_{t_o}(t) \geq \xi/c > 0$ so $(\mathcal{S}_{t_o} - L_t)(t) > 0$. Hence there is a solution in $(0, t)$. Finally if $t = |\mathbf{x}|/c$ then $t' = 0 < t$ is a solution. ■

While existence is clear geometrically, simply by drawing straight lines with (any) negative slopes and vertical axis intercept (0, t) in Fig. 12, uniqueness, on the other hand, is a bit more delicate. For there are some straight lines with negative slopes, which slopes may possibly be −1 since we cannot infer much from the figure about the horizontal (t′) and vertical scales, that intersect the curves

more than once; and for such lines the solutions to Eq. (C.4) are not unique. We now show that in fact multiple intersections do not occur.

Theorem C.2: Fix x, t according to Eq. (C.1). Then the solution to Eq. (C.4) is unique.

Proof. Suppose that L_t intersects $\mathcal{S}_{t_o}$ at two distinct points whose t′ coordinates are $t_1' < t_2'$. Clearly both t_1' and t_2' cannot be exterior to $I(t_o)$, for then L_t would have slope 0. Suppose $t_1', t_2' \in I(t_o)$. Since $\mathcal{S}_{t_o}$ is continuous on $[t_1', t_2']$ and differentiable on (t_1', t_2') then by the Mean Value Theorem there exists $t^* \in (t_1', t_2') \subseteq I^o(t_o)$ such that

$$(d\mathcal{S}_{t_o}/dt')(t^*) = [\mathcal{S}_{t_o}(t_2') - \mathcal{S}_{t_o}(t_1')]/[t_2'-t_1'] = [L_t(t_2') - L_t(t_1')]/[t_2'-t_1'] = \text{slope of } \mathcal{S}_t = -1. \quad \text{(C.7)}$$

But we have already seen from Eq. (C.5) that $|d\mathcal{S}_{t_o}/dt'| < 1$ on $I^o(t_o)$ so Eq. (C.7) yields a contradiction. Hence one of t_1', t_2' must be exterior to $I(t_o)$ and the other must be in $I(t_o)$. Suppose $t_1' < t_o$ and $t_2' \in I(t_o)$; then, since $d\mathcal{S}_{t_o}/dt'$ is continuous on $I^o(t_o)$ (see Eq. (C.5) and note that $\mathcal{S}_{t_o}(t')$ is never 0 since $\xi > 0$), we have

$$\mathcal{S}_{t_o}(t_2') = |\mathbf{x}|/c + \int_{t_o}^{t_2'} (d\mathcal{S}_{t_o}/dt')dt' \quad \text{(C.8)}$$

so

$$|\mathcal{S}_{t_o}(t_2') - |\mathbf{x}|/c| \leq \int_{t_o}^{t_2'} |d\mathcal{S}_{t_o}/dt'|dt' < t_2' - t_o. \quad \text{(C.9)}$$

But

$$|L_t(t_2') - L_t(t_1')| = t_2' - t_1' > t_2' - t_o, \quad \text{(C.10)}$$

$L_t(t_1') = |\mathbf{x}|/c$, and $L(t_2') = \mathcal{S}_{t_o}(t_2')$ so from Eqs. (C.9) and (C.10) we have

$$|\mathcal{S}_{t_o}(t_2') - |\mathbf{x}|/c| < |\mathcal{S}_{t_o}(t_2') - |\mathbf{x}|/c| \tag{C.11}$$

which again is a contradiction. Similarly, supposing $t_1' \in I(t_o)$ and $t_2' > t_{rm}(t_o)$ also leads to a contradiction. Hence L_t cannot intersect $\mathcal{S}_{t_o}$ at two distinct points and the uniqueness proof is complete. ■

We now have a precise enough description of the behavior of lines $L_t(t') = t - t'$ relative to the curves of Fig. 12 to use this figure in subsequent proofs, and we do so freely.

The next result in this section will be crucial in evaluating the integrals for $\mathbf{E}_a$ and $\mathbf{B}_a$ in Eqs. (4.69) and (4.70), giving us a simple description of the set of all $t_o \in [0, \bar{t}_o]$ such that, for given $\mathbf{x}$, t, we have $\chi_D(t_o; \bar{t}'(t_o; \mathbf{x}, t)) = 1$, thus allowing us to write Eqs. (4.73) and (4.74) for these fields. For simplicity of notation we suppress the $\mathbf{x}$ and t dependence of $\bar{t}'$ and write simply $\bar{t}'(t_o)$. Also, we introduce the following notation for the various boundaries of $D[T, G]$:

$$\partial_B D[T, G] = \bigcup_{t_o \in [0, \bar{t}_o]} \{(t_o, t_o)\}, \tag{C.12}$$

$$\partial_T D[T, G] = \bigcup_{t_o \in [0, \bar{t}_o]} \{(t_o, t_{rm}(t_o))\}, \tag{C.13}$$

$$\partial_L D[T, G] = \{0\} \times [0, \infty), \tag{C.14}$$

and

$$\partial_R D[T, G] = \{\bar{t}_o\} \times I(\bar{t}_o); \tag{C.15}$$

additionally, for future use we denote

$$\partial_{BT}D[T, G] = \partial_B D[T, G] \cup \partial_T D[T, G]. \tag{C.16}$$

Theorem C.3: Fix $\mathbf{x}$, t according to Eq. (C.1).

(i) If $-\infty < t - |\mathbf{x}|/c \le 0$ then for all $t_o \in [0, \bar{t}_o]$, $\bar{t}'(t_o) = t - |\mathbf{x}|/c$; so $(t_o, \bar{t}'(t_o)) \in D^-[T, G]$ unless $t - |\mathbf{x}|/c = 0 = t_o$ in which case $(t_o, \bar{t}'(t_o)) \in \partial_B D[T, G]$.

(ii) If $0 < t - |\mathbf{x}|/c \le \bar{t}_o$ then $(t - |\mathbf{x}|/c, t - |\mathbf{x}|/c) \in \partial_B D[T, G]$, $(t_o, \bar{t}'(t_o)) \in D^o[T, G]$ whenever $t_o \in (0, t - |\mathbf{x}|/c)$, and $(t_o, \bar{t}'(t_o)) \in D^-[T, G]$ whenever $t_o \in (t - |\mathbf{x}|/c, \bar{t}_o]$. (If $t_o = t - |\mathbf{x}|/c$ then $\bar{t}'(t_o) = t - |\mathbf{x}|/c$ so $(t_o, \bar{t}'(t_o)) \in \partial_B[T, G]$.)

(iii) If $\bar{t}_o < t - |\mathbf{x}|/c < t_{rm}(\bar{t}_o)$ then $(\bar{t}_o, t - |\mathbf{x}|/c) \in \partial_R D[T, G]$ and $(t_o, \bar{t}'(t_o)) \in D^o[T, G]$ whenever $t_o \in (0, \bar{t}_o)$; also $(\bar{t}_o, \bar{t}'(\bar{t}_o)) \in \{\bar{t}_o\} \times I^o(\bar{t}_o) \subseteq D[T, G]$ but it is never true that $\bar{t}'(t_o) = t - |\mathbf{x}|/c$.

(iv) If $t_{rm}(\bar{t}_o) \le t - |\mathbf{x}|/c < \infty$ then $[t_I(t-|\mathbf{x}|/c), t_S(t-|\mathbf{x}|/c)] \times \{t-|\mathbf{x}|/c\} \subseteq \partial_T D[T, G]$ where

$$t_I(t-|\mathbf{x}|/c) \equiv \inf\, t_{rm}^{-1}[t-|\mathbf{x}|/c], \tag{C.17}$$

$$t_S(t-|\mathbf{x}|/c) \equiv \sup\, t_{rm}^{-1}[t-|\mathbf{x}|/c], \tag{C.18}$$

and

$$t_{rm}^{-1}[t-|\mathbf{x}|/c] \equiv \{t_o \in (0, \bar{t}_o] \,|\, t_{rm}(t_o) = t-|\mathbf{x}|/c\}; \tag{C.19}$$

also $t_{rm}^{-1}[t-|\mathbf{x}|/c] \ne \varnothing$ and $t_I(t - |\mathbf{x}|/c) > 0$. In addition, $(t_o, \bar{t}'(t_o)) \in D^o[T, G]$ whenever $t_o \in (0, t_I(t - |\mathbf{x}|/c))$ and $(t_o, \bar{t}'(t_o)) \in D^-[T, G]$ whenever $t_o \in (t_S(t - |\mathbf{x}|/c), \bar{t}_o]$.

(If $t_o \in [t_I(t-|\mathbf{x}|/c), t_S(t-|\mathbf{x}|/c)]$ then $\bar{t}'(t_o) = t - |\mathbf{x}|/c$ so $(t_o, \bar{t}'(t_o)) \in \partial_T D[T, G]$.)

(v) In cases (ii) - (iv), $(0, \bar{t}'(0)) \in \partial_L D[T, G]$.

Proof. (i) It is clear from Figs. 11 and 12 (the latter augmented with the line $L_t(t') = t - t'$) that $\bar{t}'(t_o) = t - |\mathbf{x}|/c$ whenever $t - |\mathbf{x}|/c \le 0$ and $t_o \ge 0$; further, unless $t - |\mathbf{x}|/c = 0 = t_o$, then $\bar{t}'(t_o) < 0$ so from Fig. 2 we have $(t_o, \bar{t}'(t_o)) \in D^-[T, G]$, while if $t - |\mathbf{x}|/c = 0 = t_o$ then $(t_o, \bar{t}'(t_o)) = (0,0) \in \partial_B D[T, G]$.

(ii) Let $0 < t - |\mathbf{x}|/c \le \bar{t}_o$; then from Fig. 2 we have $(t - |\mathbf{x}|/c, t - |\mathbf{x}|/c) \in \partial_B D[T, G]$. Let $t_o \in (0, t - |\mathbf{x}|/c)$, i.e., $t - |\mathbf{x}|/c > t_o$; then supposing $\bar{t}'(t_o) \le t_o$ gives

$$L_t(\bar{t}'(t_o)) = t - \bar{t}'(t_o) \ge t - t_o > |\mathbf{x}|/c \tag{C.20}$$

while, from Eq. (C.4) or Fig. 12, $\mathcal{R}_{t_o}(\bar{t}'(t_o)) = |\mathbf{x}|/c$ so that $\bar{t}'(t_o)$ cannot be the retarded time for t_o, which is a contradiction; and supposing $\bar{t}'(t_o) \ge t_{rm}(t_o) = t_o + T/G(\Omega t_o)$ gives

$$L_t(\bar{t}'(t_o)) = t - \bar{t}'(t_o) \le t - [t_o + T/G(\Omega t_o)] < t - [\bar{t}_o + T/G(\Omega \bar{t}_o)] < t - \bar{t}_o \le |\mathbf{x}|/c \tag{C.21}$$

while again $\mathcal{R}_{t_o}(\bar{t}'(t_o)) = |\mathbf{x}|/c$, a contradiction. So $\bar{t}'(t_o) \in (t_o, t_{rm}(t_o)) = I^o(t_o)$ and so $(t_o, \bar{t}'(t_o)) \in D^o[T, G]$. (Alternatively, one may argue using Fig. 12 that, when $0 < t_o < t - |\mathbf{x}|/c \le \bar{t}_o$, the line $L_t(t') = t - t'$ intersects any of the curves at some $t' \in (t_o, \bar{t}_o] \subseteq (t_o, t_{rm}(t_o))$, since $\bar{t}_o < t_{rm}(t_o)$ for all $t_o \in [0, \bar{t}_o]$, so that $\bar{t}'(t_o) \in (t_o, t_{rm}(t_o))$.) Now let $t_o \in (t - |\mathbf{x}|/c, \bar{t}_o]$, i.e., $t - |\mathbf{x}|/c < t_o$; then it is clear from Fig. 12 that $\bar{t}'(t_o) = t - |\mathbf{x}|/c$ so $(t_o, \bar{t}'(t_o)) \in D^-[T, G]$.

(iii) Let $\bar{t}_o < t - |\mathbf{x}|/c < t_{rm}(\bar{t}_o)$; then from Fig. 2 we have $(\bar{t}_o, t - |\mathbf{x}|/c) \in \partial_R D[T, G]$. Let $t_o \in (0, \bar{t}_o)$; then supposing $\bar{t}'(t_o) \le t_o$ gives, similarly to Eq. (C.20),

$$L_t(\bar{t}'(t_o)) \geq t - t_o > t - \bar{t}_o > |\mathbf{x}|/c = \mathscr{S}_{t_o}(\bar{t}'(t_o)), \tag{C.22}$$

a contradiction; and supposing $\bar{t}'(t_o) \geq t_{rm}(t_o)$ gives, similarly to Eq. (C.21),

$$L_t(\bar{t}'(t_o)) < t - t_{rm}(\bar{t}_o) < t - (t - |\mathbf{x}|/c) = |\mathbf{x}|/c = \mathscr{S}_{t_o}(\bar{t}'(t_o)), \tag{C.23}$$

also a contradiction. So $(t_o, \bar{t}'(t_o)) \in D^o[T, G]$. Also, that $(\bar{t}_o, \bar{t}'(\bar{t}_o)) \in \{\bar{t}_o\} \times \Gamma(\bar{t}_o)$ may be seen from Fig. 12 by taking $t_o = \bar{t}_o$ there; note also that $\bar{t}'(\bar{t}_o) < t - |\mathbf{x}|/c$ or $\bar{t}'(t_o) > t - |\mathbf{x}|/c$ so that it is never true that $\bar{t}'(t_o) = t - |\mathbf{x}|/c$.

(iv) Let $t_{rm}(\bar{t}_o) \leq t - |\mathbf{x}|/c < \infty$; then from Fig. 2, allowing for the possibility that $t_{rm}(t_o)$ has subintervals of constancy in $(0, \bar{t}_o]$, we have $[t_I, t_S] \times \{t - |\mathbf{x}|/c\} \subseteq \partial_T D[T, G]$ where we have suppressed the argument $t - |\mathbf{x}|/c$ of t_I and t_S. Since $t_{rm}(t_o)$ increases from $t_{rm}(\bar{t}_o)$ to ∞ as t_o decreases from $\bar{t}_o$ to 0, then $t_{rm}^{-1}[t - |\mathbf{x}|/c] \neq \emptyset$ for $t_{rm}(\bar{t}_o) \leq t - |\mathbf{x}|/c < \infty$; so $t_I > 0$. Let $t_o \in (0, t_I)$; then supposing $\bar{t}'(t_o) \leq t_o$ gives, similarly to Eq. (C.20),

$$L_t(\bar{t}'(t_o)) \geq t - t_o \geq t - \bar{t}_o > t - t_{rm}(\bar{t}_o) \geq |\mathbf{x}|/c = \mathscr{S}_{t_o}(\bar{t}'(t_o)), \tag{C.24}$$

a contradiction; and supposing $\bar{t}'(t_o) \geq t_{rm}(t_o)$ and denoting by t_o^* any member of $[t_I, t_S]$, so that $t_{rm}(t_o^*) = t - |\mathbf{x}|/c$, gives

$$L_t(\bar{t}'(t_o)) < t - t_{rm}(t_o) < t - t_{rm}(t_I) \leq t - t_{rm}(t_o^*) = t - (t - |\mathbf{x}|/c) = |\mathbf{x}|/c = \mathscr{S}_{t_o}(\bar{t}'(t_o)), \tag{C.25}$$

a contradiction. So $(t_o, \bar{t}'(t_o)) \in D^o[T, G]$. Now let $t_o \in (t_S, \bar{t}_o]$; then with t_o^* as above,

$$L_t(t_{rm}(t_o)) = t - t_{rm}(t_o) > t - t_{rm}(t_S) \geq t - t_{rm}(t_o^*) = t - (t - |\mathbf{x}|/c) = |\mathbf{x}|/c \tag{C.26}$$

while $L_t(t) = 0$. But, from Eq. (C.4) we have

$$\mathscr{F}_{t_o}(t_{rm}(t_o)) = |\mathbf{x}|/c = \mathscr{F}_{t_o}(t), \tag{C.27}$$

the second equality holding because

$$t > t - |\mathbf{x}|/c = t_{rm}(t_o^*) \geq t_{rm}(t_S) > t_{rm}(t_o). \tag{C.28}$$

So $L_t(t')$ and $\mathscr{F}_{t_o}(t')$ must have their unique intersection, which occurs for $t' = \tilde{t}'(t_o)$, at a value of $t' \in (t_{rm}(t_o), t)$; i.e., $\tilde{t}'(t_o) > t_{rm}(t_o)$ and so $(t_o, \tilde{t}'(t_o)) \in D^-[T, G]$.

(v) Follows from Fig. 11 by noting that $\tilde{t}'(t_o) > 0$ whenever $t - |\mathbf{x}|/c > 0$. ■

The results are represented diagrammatically in Fig. 5, which we referred to earlier in Section IV.

The above mathematical result has, of course, physical interpretation. For example, (i) says that if the required retarded time, namely $t - |\mathbf{x}|/c$, for a charge sheet located at $z = 0$ is negative then no sheet can satisfy that requirement; this is of course clear since no sheet is in flight at any negative time. The other parts have similar interpretations; we leave their explicit elucidation to the reader.

We remark here that the accommodation in the last theorem of the possibility of intervals of constancy of t_{rm} is not unwarranted generalization -- we presented in Appendix A a class of pulses for which t_{rm} has such an interval of constancy.

Theorem C.3 tells us that the map $t_o \mapsto \chi_D(t_o; \tilde{t}'(t_o; \mathbf{x}, t))$, for fixed $\mathbf{x}$, t, is piecewise continuous on $[0, \tilde{t}_o]$, being given by

$$\chi_D(t_o;\bar{t}'(t_o;\mathbf{x},t)) = \begin{cases} 1, & \text{if } t_o \in [0,\overline{T}_o(\mathbf{x},t)] \\ \\ 0, & \text{if } t_o \in (\overline{T}_o(\mathbf{x},t),\bar{t}_o] \ , \end{cases} \tag{C.29}$$

where $\overline{T}_o(\mathbf{x}, t)$ is given by

$$\overline{T}_o(\mathbf{x},t) \equiv \begin{cases} 0, & \text{if } -\infty < t-|\mathbf{x}|/c \le 0 \\ t-|\mathbf{x}|/c, & \text{if } 0 < t-|\mathbf{x}|/c \le \bar{t}_o \\ \bar{t}_o, & \text{if } \bar{t}_o < t-|\mathbf{x}|/c < t_{rm}(\bar{t}_o) \\ t_f(t-|\mathbf{x}|/c), & \text{if } t_{rm}(\bar{t}_o) \le t-|\mathbf{x}|/c < \infty \ . \end{cases} \tag{C.30}$$

Hence the χ_D factors in Eqs. (4.69) and (4.70) can be accounted for simply by choosing 0 for the integration lower limit and $\overline{T}_o(\mathbf{x}, t)$ for the upper limit and setting $\chi_D \equiv 1$. In Fig. 6 we plot $\overline{T}_o(\mathbf{x}, t)$ as a function of t for fixed $\mathbf{x}$, with one jump corresponding to one interval of constancy for $t_{rm}(t_o)$, as in Fig. 5; recall that we also referred to Fig. 6 earlier, in Section IV.

Strictly speaking, the upper limit for the last case of Eq. (C.30) should be $t_S(t - |\mathbf{x}|/c)$ (see Theorem C.3(iv)). We have chosen t_I rather than t_S because the choice of t_I allows a more unified (i.e., caseless) treatment in the sequel than does the choice of t_S. This point is almost always moot since in almost all non-contrived cases we have $t_I = t_S$ (the pulses of Appendix A are examples of "contrived" pulses). Nevertheless, we will be careful to point out and include those modifications to our formulation that are necessary in case $t_S > t_I$. As a point of interest, note that the choice of t_S rather than t_I for the definition of $\overline{T}_o$ would make the map $t \mapsto \overline{T}_o(\mathbf{x}, t)$ left-continuous rather than right-continuous at the jump in Fig. 6.

The last two results of this Appendix establish the continuity and differentiability properties of the map $t_o \mapsto \bar{t}'(t_o; \mathbf{x}, t)$, with $\mathbf{x}$, t fixed and t_o variable according to Eq. (C.1), that we will need in Appendix D. Consistent with our aforementioned convention of suppressing $\mathbf{x}$ and t in the argument list of $\bar{t}'$, we also write $(d\bar{t}'/dt_o)(t_o)$ to mean $(\partial\bar{t}'/\partial t_o)(t_o; \mathbf{x}, t)$. We denote

$$D_e[T, G] \equiv [0, \bar{t}_o] \times (-\infty, \infty), \tag{C.31}$$

with interior given by

$$D_e^o[T, G] = (0, \bar{t}_o) \times (-\infty, \infty); \tag{C.32}$$

and we also denote

$$(D_e^o)_b[T, G] \equiv D_e^o[T, G] \setminus \partial_{BT} D[T, G] = D^o[T, G] \cup D^{-o}[T, G] \tag{C.33}$$

(see Eq. (C.16)).

Theorem C.4: Fix $\mathbf{x}$, t according to Eq. (C.1).

(i) If $-\infty < t - |\mathbf{x}|/c \le 0$ then $\tilde{t}'(t_o)$ is continuously differentiable (C^1) on $[0, \bar{t}_o]$.

(ii) If $0 < t - |\mathbf{x}|/c \le \bar{t}_o$ then $\tilde{t}'(t_o)$ is C^1 on $(0, t - |\mathbf{x}|/c) \cup (t - |\mathbf{x}|/c, \bar{t}_o)$.

(iii) If $\bar{t}_o < t - |\mathbf{x}|/c < t_{rm}(\bar{t}_o\ )$ then $\tilde{t}(t_o)$ is C^1 on $(0, \bar{t}_o)$.

(iv) If $t_{rm}(\bar{t}_o) \le t - |\mathbf{x}|/c < \infty$ then $\tilde{t}'(t_o)$ is C^1 on

$(0, t_1(t-|\mathbf{x}|/c)) \cup (t_1(t-|\mathbf{x}|/c), t_S(t-|\mathbf{x}|/c)) \cup (t_S(t-|\mathbf{x}|/c), \bar{t}_o)$.

Further, for those t_o at which $\tilde{t}'(t_o)$ is C^1 we have in each case

$$(d\tilde{t}'/dt_o)(t_o) = \begin{cases} -(1/c)\Delta(t_o)(d\mathcal{B}/dt_o)(t_o), & \text{if } t_o \in \text{FI [a first interval of (i)–(iv)]} \\ 0, & \text{if } t_o \notin \text{FI [not a first interval of (i)–(iv)]} \end{cases} \tag{C.34}$$

and

$$(d\bar{t}'/dt_o)(t_o) = \begin{cases} -(1/c)\Delta(t_o)(\partial Z/\partial t_o)(t_o;\bar{t}'(t_o))/[1+(1/c)\Delta(t_o)(\partial Z/\partial\tau)(t_o;\bar{t}'(t_o))], & \text{if } t_o \in FI \\ 0, & \text{if } t_o \notin FI \end{cases} \tag{C.35}$$

where

$$\mathcal{S}(t_o) = Z(t_o;\bar{t}'(t_o)), \tag{C.36}$$

or, more precisely,

$$\mathcal{S}(t_o;\mathbf{x},t) = Z(t_o;\bar{t}'(t_o;\mathbf{x},t)), \tag{C.37}$$

and

$$\Delta(t_o) = [\mathcal{S}(t_o) - \zeta]/\{\xi^2 + [\mathcal{S}(t_o)-\zeta]^2\}^{1/2} \tag{C.38}$$

(and the arguments of Z are denoted (t_o, τ)).

Proof. The proof of (i) here follows immediately from (i) of Theorem C.3, for we have $d\bar{t}'/dt_o = 0$ on $[0, \bar{t}_o]$. To prove (ii)-(iv) we appeal to the Implicit Function Theorem which says that if real-valued function $\mathcal{H}(\tau_o, \tau')$ has continuous first partial derivatives $\partial\mathcal{H}/\partial\tau_o$ and $\partial\mathcal{H}/\partial\tau'$ in some open subset of $\mathbb{R}^2$ which contains interior point (t_o, t_o') satisfying $\mathcal{H}(t_o, t_o') = 0$ and $(\partial\mathcal{H}/\partial\tau')(t_o, t_o') \neq 0$, then there is some nonvoid neighborhood $(t_o - \delta, t_o + \delta)$ of t_o and a unique function h defined on $(t_o - \delta, t_o + \delta)$ such that $t_o' = h(t_o)$ and $\mathcal{H}(\tau_o, h(\tau_o)) = 0$ whenever $\tau_o \in (t_o - \delta, t_o + \delta)$; further, h is C^1 on $(t_o - \delta, t_o + \delta)$. To apply this to our present situation we first define, for
$(t_o, t') \in D^o_{*}[T, G]$,

$$\mathcal{H}(t_o, t') = t' - t + \mathcal{T}_{t_o}(t'). \tag{C.39}$$

We next note, using a rewritten version of the RHS of Eq. (C.4), that

$$\mathcal{T}_{t_o}(t') = \begin{cases} (1/c)\{\xi^2 + [Z(t_o; t') - \zeta]^2\}^{1/2}, & \text{if } t' \in I(t_o) \\ |\mathbf{x}|/c, & \text{if } t' \in I^-(t_o) \end{cases} \tag{C.40}$$

and, from the differentiability properties of $Z(t_o; t)$ on $D_e[T, G]$ (see Eq. (3.34)), that $\partial\mathcal{H}/\partial t_o$ and $\partial\mathcal{H}/\partial t'$ exist and are continuous on $(D_e^\circ)_b[T, G]$ with

$$(\partial\mathcal{H}/\partial t')(t_o, t') = \begin{cases} 1 + (d\mathcal{T}_{t_o}/dt')(t'), & \text{if } t' \in I^\circ(t_o) \\ 1, & \text{if } t' \in I^-(t_o) \end{cases} \tag{C.41}$$

there; and we then use Eq. (C.5) to conclude that $\partial\mathcal{H}/\partial t' \neq 0$ there. Now if $t_o \in (0, \bar{t}_o)$ then our previous existence and uniqueness results imply that $(t_o, \bar{t}'(t_o)) \in D_e^\circ[T, G]$ is such that

$$\mathcal{H}(t_o, \bar{t}'(t_o)) = 0 \text{ for all } t_o \in (0, \bar{t}_o) \tag{C.42}$$

(not just t_o–locally) so $\bar{t}'$ must be the unique h guaranteed by the Implicit Function Theorem whenever

$$(t_o, \bar{t}'(t_o)) \in (D_e^\circ)_b\,[T, G]\,; \tag{C.43}$$

hence $\bar{t}'$ must be C^1 at such t_o. But items (ii)-(iv) of Theorem C.3 indicate that Eq. (C.43) is satisfied precisely for those t_o specified in cases (ii)-(iii) here, and, in case (iv) here, for

$t_o \in (0, t_l) \cup (t_s, \bar{t}_o)$. Since, in case (iv), $\bar{t}'$ is in fact constant on $[t_l, t_s]$ (being $t - |\mathbf{x}|/c$ there) then we may include the interval (t_l, t_s) so that (iv) is also proved. Lastly, the first expression for $d\bar{t}'/dt_o$ results from differentiating

$$\mathcal{H}(t_o, \bar{t}'(t_o)) = \bar{t}'(t_o) - t + \mathcal{F}_{t_o}(\bar{t}'(t_o)) \tag{C.44}$$

(see Eq. (C.39)) with respect to t_o, using Eq. (C.42); and the second expression for $d\bar{t}'/dt_o$ results from noting that

$$(d\mathcal{G}/dt_o)(t_o) = (\partial Z/\partial t_o)(t_o; \bar{t}'(t_o)) + (\partial Z/\partial \tau)(t_o; \bar{t}'(t_o))(d\bar{t}'/dt_o)(t_o). \tag{C.45}$$

In both cases, the continuous differentiability of Z on $(D^{\circ}_{\infty})_b[T, G]$ and of $\bar{t}'$ on the intervals specified in (i)-(iv) guarantee the existence of (continuous) $d\mathcal{G}/dt_o$ on those same intervals. ■

The above theorem indicates that there may be exceptional points in $[0, \bar{t}_o]$ where $\bar{t}'$ fails to be C^1. The following is true however.

Theorem C.5: Fix **x**, t according to Eq. (C.1). Then $\bar{t}'$ is continuous on $[0, \bar{t}_o]$.

Proof. If $-\infty < t - |\mathbf{x}|/c \leq 0$ then the result follows immediately from Theorem C.4(i). So let $0 < t - |\mathbf{x}|/c < \infty$ and consider first continuity at $t_o = 0$.

Define

$$t_o^{<} = \min\,\{t - |\mathbf{x}|/c,\ t_1(t - |\mathbf{x}|/c), \bar{t}_o\}; \tag{C.46}$$

by items (ii)-(iv) of Theorem C.3, it follows that $\bar{t}'(t_o) \in \Gamma^o(t_o)$ for all $t_o \in (0, t_o^<)$ and so for such t_o, $\mathcal{S}_{t_o}(\bar{t}'(t_o))$ is given by the first line of Eq. (C.40). We claim -- and show below -- that $\lim_{t_o \to 0^+} \bar{t}'(t_o)$ exists; then $\lim_{t_o \to 0^+} \mathcal{S}_{t_o}(\bar{t}'(t_o))$ exists and is given by

$$\lim_{t_o \to 0^+} \mathcal{S}_{t_o}(\bar{t}'(t_o)) = (1/c)\{\xi^2 + [v_o \lim_{t_o \to 0^+} \bar{t}'(t_o) - \zeta]^2\}^{1/2}\ , \tag{C.47}$$

since $\lim_{t_o \to 0^+} G(\Omega t_o) = 0$ and the square and square root are continuous on non-negative reals. Hence from Eq. (C.44) we have, using (from Eq. (C.42)) $\lim_{t_o \to 0^+} \mathcal{H}(t_o, \bar{t}'(t_o)) = 0$, that

$$t - \lim_{t_o \to 0^+} \bar{t}'(t_o) = (1/c)\{\xi^2 + [v_o \lim_{t_o \to 0+} \bar{t}'(t_o) - \zeta]^2\}^{1/2}, \tag{C.48}$$

where $\bar{t}'(t_o) > t_o \geq 0$. But this last equation is the same as that of (the first line of) Eq. (C.2) and we know the latter has unique solution $\bar{t}'(0)$ given by Eq. (C.3). So we must have

$$\lim_{t_o \to 0^+} \bar{t}'(t_o) = \bar{t}'(0), \tag{C.49}$$

i.e., $\bar{t}'(t_o)$ is continuous at $t_o = 0$.

To show that $\lim_{t_o \to 0^+} \bar{t}'(t_o)$ indeed exists we proceed as follows, still requiring $t_o \in (0, t_o^<)$. In Eq. (C.45) for $d\mathcal{S}/dt_o$ we substitute the first line of Eq. (C.34) for $d\bar{t}'/dt_o$ to get

$$(d\mathcal{S}/dt_o)(t_o) = (\partial Z/\partial t_o)(t_o; \tilde{t}'(t_o)) \,/\, [1 + (1/c)\Delta(t_o)(\partial Z/\partial \tau)(t_o; \tilde{t}'(t_o))] \tag{C.50}$$

where we can be sure that the denominator on the RHS is strictly positive since, by Eqs. (C.38) and (3.35),

$$|\Delta(t_o)| < 1 \quad \text{and} \quad |(\partial Z/\partial \tau)(t_o; \tilde{t}'(t_o))| \leq v_o . \tag{C.51}$$

But $(\partial Z/\partial t_o)(t_o; t) < 0$ for $(t_o, t) \in D^{\circ}[T, G]$, as per Corollary B.1, so that $(\partial Z/\partial t_o)(t_o; \tilde{t}'(t_o)) < 0$; hence

$$(d\mathcal{S}/dt_o)(t_o) < 0, \qquad t_o \in (0, t_o^{<}), \tag{C.52}$$

i.e., $\mathcal{S}$ is strictly decreasing on $(0, t_o^{<})$. And by Eq. (C.34),

$$(d\tilde{t}'/dt_o)(t_o) = (1/c)|(d\mathcal{S}/dt_o)(t_o)|\Delta(t_o), \qquad t_o \in (0, t_o^{<}). \tag{C.53}$$

Now if $s \equiv \sup_{t_o \in (0, t_o^{<})} \mathcal{S}(t_o) \leq \zeta$ then $\Delta(t_o) \leq 0$ on $(0, t_o^{<})$ so that

$$(d\tilde{t}'/dt_o)(t_o) \leq 0, \quad s \leq \zeta, \quad t_o \in (0, t_o^{<}) \tag{C.54}$$

while if $s > \zeta$ then, because $\mathcal{S}$ is strictly decreasing on $(0, t_o^{<})$, there exists $t_o^{<<} \in (0, t_o^{<})$ such that

$\Delta(t_o) > 0$ on $(0, t_o^{<<})$ so that

$$(d\tilde{t}'/dt_o)(t_o) > 0, \quad s > \zeta, \quad t_o \in (0, t_o^{<<}). \tag{C.55}$$

Further, since $0 < t_o < \bar{t}'(t_o) < t$ on $(0, t_o^{<})$, then in either case $\bar{t}'(t_o)$ is bounded and monotonic on $(0, t_o^{<<})$ so that $\lim_{t_o \to 0^+} \bar{t}'(t_o)$ exists and our claim is proved.

Lastly, we show continuity at the other exceptional points. To that end, let $t_o^{>}$ be defined by

$$t_o^{>} \equiv \begin{cases} t - |\mathbf{x}|/c, & \text{if } 0 < t - |\mathbf{x}|/c \le \bar{t}_o \\ \bar{t}_o, & \text{if } \bar{t}_o < t - |\mathbf{x}|/c < t_m(\bar{t}_o) \\ t_1(t - |\mathbf{x}|/c), & \text{if } t_m(\bar{t}_o) \le t - |\mathbf{x}|/c < \infty; \end{cases} \tag{C.56}$$

then, for $t_o \in (0, t_o^{>})$, Eqs. (C.52) and (C.53) still hold. Now if $i \equiv \inf_{t_o \in (0, t_o^{>})} \mathcal{S}(t_o) \ge \zeta$ then $\Delta(t_o) \ge 0$ on $(0, t_o^{>})$ so that

$$(d\tilde{t}'/dt_o)(t_o) \ge 0, \quad i \ge \zeta \qquad t_o \in (0, t_o^{>}) \tag{C.57}$$

while if $i < \zeta$ then there exists $t_o^{><} \in (0, t_o^{>})$ such that $\Delta(t_o) < 0$ on $(t_o^{><}, t_o^{>})$ so that

$$(d\tilde{t}'/dt_o)(t_o) < 0, \quad i < \zeta, \quad t_o \in (t_o^{><}, t_o^{>}). \tag{C.58}$$

Hence $\bar{t}'(t_o)$ is bounded and monotonic on $(t_o^{><}, t_o^{>})$ so $\lim_{t_o \to (t_o^{>})^-} \bar{t}'(t_o)$ exists. Then from Eq. (C.44) we have

$$t - \lim_{t_o \to (t_o^{>})^-} \bar{t}'(t_o) = (1/c)\left(\xi^2 + \{v_o[\lim_{t_o \to (t_o^{>})^-} \bar{t}'(t_o) - t_o^{>}] - (v_o/T)G(\Omega t_o^{>})[\lim_{t_o \to (t_o^{>})^-} \bar{t}'(t_o) - t_o^{>}]^2 - \zeta\}^2\right)^{1/2} \quad \text{(C.59)}$$

so by uniqueness of solution to this equation (which is the first line of Eq. (C.4)) we have

$$\lim_{t_o \to (t_o^{>})^-} \bar{t}'(t_o) = \bar{t}'(t_o^{>}). \quad \text{(C.60)}$$

This is the continuity result in case $t_o^{>} = \bar{t}_o$; otherwise, we have from Theorem C.3 that

$$\lim_{t_o \to (t_o^{>})^+} \bar{t}'(t_o) = t - |\mathbf{x}|/c = \bar{t}'(t_o^{>}) \quad \text{(C.61)}$$

so

$$\lim_{t_o \to t_o^{>}} \bar{t}'(t_o) = \bar{t}'(t_o^{>}) = t - |\mathbf{x}|/c \quad \text{(C.62)}$$

which is the continuity result for the other two values of $t_o^{>}$ (as well as for $t_o^{>} = t_S(t - |\mathbf{x}|/c)$). Finally, continuity at $t_o = \bar{t}_o$ in cases $t - |\mathbf{x}|/c \le \bar{t}_o$ and $t - |\mathbf{x}|/c \ge t_m(\bar{t}_o)$ follows from the fact that $\bar{t}'(t_o) = t - |\mathbf{x}|/c$ in some (one-sided) non-punctured neighborhood of $\bar{t}_o$. ■

Corollary C.6: Fix $\mathbf{x}$, t according to Eq. (C.1). Then the map $t_o \mapsto \mathcal{S}(t_o; \mathbf{x}, t)$ is strictly decreasing on $[0, \overline{T}_o(\mathbf{x}, t)]$.

Proof. From Eq. (C.52)ff, $d\mathcal{S}/dt_o < 0$ on $(0, \overline{T}_o(\mathbf{x}, t))$; and from Eq. (C.37) and Theorem C.5, the indicated map is continuous on $[0, \overline{T}_o(\mathbf{x}, t)]$. ■

APPENDIX D: PROOF OF THEOREM 5.1

We present our work as a sequence of five results culminating in existence Theorem 5.1. We will use in the sequel Λ_* rewritten in the form

$$\Lambda_*(t_o; t) = (1/\sin\phi)|x|^{-1}\, Z(t_o; t) - \cot\phi. \tag{D.1}$$

Lemma D.1: Let a, b, c_1, c_2, d_1, d_2, $c' \in \mathbb{R}$, $a < b$, $c_i < d_i$ $(i = 1, 2)$, $c' > 0$; and $M_i \equiv \max\{|c_i|, |d_i|\} > 0$. Let functions h: $[a, b] \times [c_1, d_1] \rightarrow \mathbb{R}$ and k_i: $[a, b] \times [c', \infty) \rightarrow [c_i, d_i]$ be such that

(i) h is continuous, with $\| h \|_\infty \equiv M < \infty$;

(ii) $\forall_{y \in [c', \infty)}$ $(t_o; y) \mapsto k_i(t_o; y)$ is continuous, $i = 1, 2$; and

(iii) $\forall_{t_o \in [a, b]}$ $\lim_{y \to \infty} k_i(t_o; y) = k_{i,\infty}(t_o) \in [c_i, d_i]$, $i = 1, 2$, uniformly on [a, b].

Further, for $y \in [a, b]$ define

$$\mathcal{H}(y) = \int_a^b dt_o\, k_2(t_o; y) h(t_o, k_1(t_o; y)). \tag{D.2}$$

Then

$$\lim_{y \to \infty} \mathcal{H}(y) = \int_a^b dt_o\, k_{2,\infty}(t_o) h(t_o, k_{1,\infty}(t_o)). \tag{D.3}$$

Proof: Let $\varepsilon > 0$. Since h is uniformly continuous on $[a, b] \times [c_1, d_1]$, there exists $\delta > 0$ such that $\forall_{t_o \in [a, b]}$ $|k - k'| < \delta \rightarrow |h(t_o, k) - h(t_o, k')| < \varepsilon/2M_2(b - a)$. Since the convergence of $k_i(t_o; y)$ to $k_{i,\infty}(t_o)$ is uniform, then $\exists_{Y > 0}$ $\forall_{t_o \in [a, b]}$ $y > Y \rightarrow$ $|k_1(t_o; y) - k_{1,\infty}(t_o)| < \delta$ and $|k_2(t_o; y) - k_{2,\infty}(t_o)| < \varepsilon/2(M + 1)(b - a)$; further, $k_{1,\infty}(t_o)$ is continuous, hence integrable, on [a, b].

Now if y > Y then $\forall_{t_o \in [a,b]}$

$$|k_{2,\infty}(t_o)h(t_o, k_{1,\infty}(t_o)) - k_2(t_o; y)\, h(t_o, k_1(t_o; y))|$$

$$\le |h(t_o, k_{1,\infty}(t_o))| \cdot |k_{2,\infty}(t_o) - k_2(t_o; y)| + |k_2(t_o; y)| \cdot |h(t_o, k_{1,\infty}(t_o)) - h(t_o, k_1(t_o; y))|$$

$$\le (M + 1) \cdot \varepsilon / 2(M + 1)(b - a) + M_2 \cdot \varepsilon / 2M_2(b - a) = \varepsilon/(b - a) \tag{D.4}$$

and so

$$\int_a^b dt_o |k_{2,\infty}(t_o)h(t_o, k_{1,\infty}(t_o)) - k_2(t_o; y)h(t_o, k_1(t_o; y))| < \varepsilon \tag{D.5}$$

whenever y > Y. ■

Proposition D.2: Let $\hat{\mathbf{x}} = (\sin\phi, 0, \cos\phi)$ with $0 \le \phi \le \pi/2$, $\tau > 0$, and $t_o \in [0, \bar{T}_o^{-}(\tau)]$.

Then

$$\lim_{|\mathbf{x}| \to \infty} |\mathbf{x}|^{-1} Z(t_o; \tilde{t}\,'(t_o; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau)) = 0 \tag{D.6}$$

t_0-uniformly on $[0, \bar{T}_o^{-}(\tau)]$.

Proof: From the first line of Equation (C.3), which applies here since $|\mathbf{x}|/c + \tau > |\mathbf{x}|/c$, we compute

$$\lim_{|\mathbf{x}| \to \infty} \tilde{t}\,'(0; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau) = \tau/[1 - (v_o/c)\cos\phi]. \tag{D.7}$$

Since

$$Z(0; \bar{t}'(0; |x|\hat{x}, |x|/c + \tau)) = v_o \bar{t}'(0; |x|\hat{x}, |x|/c + \tau) \tag{D.8}$$

then

$$\lim_{|x|\to\infty} |x|^{-1} Z(0; \bar{t}'(0; |x|\hat{x}, |x|/c + \tau)) = 0. \tag{D.9}$$

But by Corollary C.6,

$$0 \le Z(t_o; \bar{t}'(t_o; |x|\hat{x}, |x|/c + \tau)) \le Z(0; \bar{t}'(0; |x|\hat{x}, |x|/c + \tau)) \tag{D.10}$$

for all $t_o \in [0, \bar{T}_o^{\infty}(\tau)]$, since $\bar{T}_o(|x|\hat{x}, |x|/c + \tau) = \bar{T}_o^{\infty}(\tau)$. Hence, we are done. ■

Proposition D.3: Let $\hat{x} = (\sin\phi, 0, \cos\phi)$ with $0 < \phi \le \pi/2$, $\tau > 0$, and $t_o \in [0, \bar{T}_o^{\infty}(\tau)]$.

Then

(i) we have

$$\lim_{|x|\to\infty} \bar{t}'(t_o; |x|\hat{x}, |x|/c + \tau) = \bar{t}'_{\infty}(t_o; \phi, \tau) \tag{D.11}$$

t_0-pointwise on $[0, \bar{T}_o^{\infty}(\tau)]$, where

$$\bar{t}'_-(t_o;\phi,\tau) = \begin{cases} \tau/C_1(\phi), & \text{if } t_o = 0 \\ \tau, & \text{if } t_o \neq 0 \text{ and } \phi = \pi/2 \\ \tau - [1/C_2(\phi)G(\Omega t_o)]\{C_1(\phi) & \\ + C_2(\phi)G(\Omega t_o)(\tau - t_o) & \\ - [C_1^2(\phi) + 2C_2(\phi)G(\Omega t_o)(\tau - t_o)]^{1/2}\}, & \text{if } t_o \neq 0 \text{ and } 0 < \phi < \pi/2 \end{cases}$$

(D.12)

for

$$C_1(\phi) = 1 - (v_o/c)\cos\phi \quad \text{and} \quad C_2(\phi) = 2T^{-1}(v_o/c)\cos\phi\,; \tag{D.13}$$

(ii) $t_0 \mapsto \bar{t}'_-(t_0;\phi,\tau)$ is continuous on $[0, \bar{T}_o^-(\tau)]$;

(iii) the convergence in (i) is actually t_0-uniform on $[0, \bar{T}_o^-(\tau)]$; and

(iv) $\lim_{\phi \to (\pi/2)^-} \bar{t}'_-(t_o;\phi,\tau) = \tau$ for all $t_o \in [0, \bar{T}_o^-(\tau)]$.

Proof: (i) If $t_o = 0$, then the result follows immediately from Equations (D.7) and (D.13). so suppose $t_o > 0$. Since $\tau > 0$, then Equation (5.16) holds for all $|\mathbf{x}| > 0$ so that, from Equation (C.4) with $\mathbf{x} = |\mathbf{x}|\hat{\mathbf{x}}$ and $t = |\mathbf{x}|/c + \tau$, we have for all $|\mathbf{x}| > 0$

$$\begin{aligned} &|\mathbf{x}|/c + \tau - \bar{t}'(t_o; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau) \\ &= (|\mathbf{x}|/c)\{\sin^2\phi + [|\mathbf{x}|^{-1}Z(t_o; \bar{t}'(t_o; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau)) - \cos^2\phi]^2\}^{1/2}. \end{aligned} \tag{D.14}$$

Squaring both sides of this equation and using $\bar{t}' - t_o = (\bar{t}' - \tau) + (\tau - t_o)$ yields

$$
\begin{aligned}
&C_2(\phi)G(\Omega t_o)[\bar{t}'(t_o; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau) - \tau]^2 \\
&+ 2[C_1(\phi) + C_2(\phi)G(\Omega t_o)(\tau - t_o)][\bar{t}'(t_o; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau) - \tau] \\
&- (\tau - t_o)[2 - 2C_1(\phi) - C_2(\phi)G(\Omega t_o)(\tau - t_o)] \\
&= D_{34}(t_o; |\mathbf{x}|) + D_2(t_o; |\mathbf{x}|)[\bar{t}'(t_o; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau) - \tau]^2 \\
&+ D_1(t_o; |\mathbf{x}|)[\bar{t}'(t_o; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau) - \tau] + D_0(t_o; |\mathbf{x}|)
\end{aligned}
\tag{D.15}
$$

where $C_1(\phi)$ and $C_2(\phi)$ are given by Eq. (D.13) and

$$
\begin{aligned}
D_{34}(t_o; |\mathbf{x}|) = &-(|\mathbf{x}|/c)^{-1}(v_o/c)^2 T^{-1}G(\Omega t_o)[\bar{t}'(t_o; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau) - \tau]^3 \\
&\times \{T^{-1}G(\Omega t_o)[\bar{t}'(t_o; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau) - \tau] + 2[2T^{-1}G(\Omega t_o)(\tau - t_o) - 1]\},
\end{aligned}
\tag{D.16}
$$

$$
D_2(t_o; |\mathbf{x}|) = (|\mathbf{x}|/c)^{-1}\left(1 - (v_o/c)^2 \{1 - 6T^{-1}G(\Omega t_o)(\tau - t_o) + 6[T^{-1}G(\Omega t_o)(\tau - t_o)]^2\}\right), \tag{D.17}
$$

$$
D_1(t_o; |\mathbf{x}|) = -2(|\mathbf{x}|/c)^{-1}(v_o/c)^2(\tau - t_o)\{1 - 3T^{-1}G(\Omega t_o)(\tau - t_o) + 2[T^{-1}G(\Omega t_o)(\tau - t_o)]^2\}, \tag{D.18}
$$

and

$$
D_0(t_o; |\mathbf{x}|) = -(|\mathbf{x}|/c)^{-1}(v_o/c)^2(\tau - t_o)^2[1 - T^{-1}G(\Omega t_o)(\tau - t_o)]^2 ; \tag{D.19}
$$

note that we have supressed the τ-dependence of D_0, D_1, D_2, and D_{34} and the $\hat{\mathbf{x}}$-dependence of D_{34} in their argument lists. In the above we have introduced extraneous roots for $\bar{t}'$ in the squaring process

but, by Theorems C.1 and C.2, Eq. (D.14) has unique solution for all $|\mathbf{x}| > 0$ and $\tau \geq 0$. Now since $t_o \in [0, \bar{T}_o^-(\tau)]$ then $\tau \in [t_o, t_o + T/G(\Omega t_o)]$; combining this with Eq. (5.16) we then have for all $|\mathbf{x}| > 0$

$$|\bar{t}'(t_o; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau) - \tau| < T/G(\Omega t_o). \tag{D.20}$$

Thus

$$|D_{34}(t_o; |\mathbf{x}|)| \leq 7(|\mathbf{x}|/c)^{-1} (v_o/c)^2 \, T^2/G^2(\Omega t_o), \tag{D.21}$$

$$|D_2(t_o; |\mathbf{x}|)| \cdot |\bar{t}'(t_o; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau) - \tau|^2 \leq (|\mathbf{x}|/c)^{-1}[1 + 13(v_o/c)^2] T^2/G^2(\Omega t_o), \tag{D.22}$$

$$|D_1(t_o; |\mathbf{x}|)| \cdot |\bar{t}'(t_o; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau) - \tau| \leq 12(|\mathbf{x}|/c)^{-1}(v_o/c)^2 T^2/G^2(\Omega t_o), \tag{D.23}$$

and

$$|D_0(t_o; |\mathbf{x}|)| \leq 4(|\mathbf{x}|/c)^{-1}(v_o/c)^2] T^2/G^2(\Omega t_o) \tag{D.24}$$

so that

$$|\text{RHS of Eq. (D.15)}| \leq [1 + 36(v_o/c)^2](|\mathbf{x}|/c)^{-1} T^2/G^2(\Omega t_o) < 2(|\mathbf{x}|/c)^{-1} T^2/G^2(\Omega t_o). \tag{D.25}$$

If $\phi = \pi/2$ then from Eqs. (D.15) and (D.25) we have

$$|\tilde{t}'(t_o; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau) - \tau| \leq (|\mathbf{x}|/c)^{-1} T^2/G^2(\Omega t_o) \tag{D.26}$$

so

$$\lim_{|\mathbf{x}|\to\infty} \tilde{t}'(t_o; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau) = \tau \qquad (\phi = \pi/2) \tag{D.27}$$

t_0-pointwise on $(0, \tilde{T}_o^-(\tau)]$. Since this result must hold for all roots of Eq. (D.15), it also holds in particular for the unique root of Eq. (D.14), so the result stated for $\phi = \pi/2$ in Eq. (D.12) follows.

Suppose then that $0 < \phi < \pi/2$ and consider the quadratic equation *LHS* of Eq. *(D.15) = 0, i.e.,*

$$C_2(\phi)G(\Omega t_o)w^2 + 2[C_1(\phi) + C_2(\phi)G(\Omega t_o)(\tau - t_o)]w - (\tau - t_o)[2 - 2C_1(\phi) - C_2(\phi)G(\Omega t_o)(\tau - t_o)] = 0 \tag{D.28}$$

with solutions

$$w_\mp^\pm(t_o;\phi,\tau) \equiv -[1/C_2(\phi)G(\Omega t_o)]\{C_1(\phi) + C_2(\phi)G(\Omega t_o)(\tau - t_o) \mp [C_1^2(\phi) + 2C_2(\phi)G(\Omega t_o)(\tau - t_o)]^{1/2}\}. \tag{D.29}$$

Subtracting Eq. (D.28), with w there replaced by $w_\mp^\pm(t_o;\phi,\tau)$, from Eq. (D.15), with notation

$$w(t_o; |\mathbf{x}|, \phi, \tau) \equiv \tilde{t}'(t_o; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau) - \tau \tag{D.30}$$

in the latter equation, we find, using Eq. (D.25), that for all $|\mathbf{x}| > 0$

$$|[w(t_o;\,|\mathbf{x}|,\phi,\tau) + Q(t_o,\phi,\tau)]^2 - [w_-^{\pm}(t_o;\phi,\tau) + Q(t_o,\phi,\tau)]^2| \le 2(|\mathbf{x}|/c)^{-1}[1/C_2(\phi)G(\Omega t_o)]T^2/G^2(\Omega t_c) \tag{D.31}$$

where

$$Q(t_o,\phi,\tau) \equiv [C_1(\phi) + C_2(\phi)G(\Omega t_o)(\tau - t_o)]/C_2(\phi)G(\Omega t_o). \tag{D.32}$$

Hence there are four possibilities:

$$\lim_{|\mathbf{x}|\to\infty} w(t_o;\,|\mathbf{x}|,\phi,\tau) = w_-^{+}(t_o;\phi,\tau), \tag{D.33}$$

$$\lim_{|\mathbf{x}|\to\infty} w(t_o;\,|\mathbf{x}|,\phi,\tau) = w_-^{-}(t_o;\phi,\tau), \tag{D.34}$$

$$\lim_{|\mathbf{x}|\to\infty} w(t_o;\,|\mathbf{x}|,\phi,\tau) = -[w_-^{+}(t_o;\phi,\tau) + 2Q(t_o,\phi,\tau)], \tag{D.35}$$

and

$$\lim_{|\mathbf{x}|\to\infty} w(t_o;\,|\mathbf{x}|,\phi,\tau) = -[w_-^{-}(t_o;\phi,\tau) + 2Q(t_o,\phi,\tau)], \tag{D.36}$$

corresponding to the four roots of Eq. (D.15). The correct choice is easily determined by noting that

$$\tau + \lim_{t_o\to 0^+} w_-^{-}(t_o;\phi,\tau) = -\tau/C_1(\phi) = \tau + \lim_{t_o\to 0^+}\{-[w_-^{\pm}(t_o;\phi,\tau) + 2Q(t_o,\phi,\tau)]\} \tag{D.37}$$

so that these three choices for the limit yield (for $\tau > 0$)

$$\lim_{t_o \to 0^+} \lim_{|\mathbf{x}| \mapsto \infty} \tilde{t}'(t_o;\ |\mathbf{x}|\hat{\mathbf{x}},\ |\mathbf{x}|/c + \tau) = \tau + \lim_{t_o \to 0^+} w(t_o;\ |\mathbf{x}|, \phi, \tau) = -\tau/C_1(\phi) < 0 \tag{D.38}$$

yet, by Eq. (5.16), the iterated limit must be $\geq t_o \geq 0$. On the other hand,

$$\tau + \lim_{t_o \to 0^+} w_-^+(t_o;\phi,\tau) = \tau/C_1(\phi); \tag{D.39}$$

hence the correct choice is Eq. (D.33) and so

$$\lim_{|\mathbf{x}| \mapsto \infty} \tilde{t}'(t_o;\ |\mathbf{x}|\hat{\mathbf{x}},\ |\mathbf{x}|/c + \tau) = \tau + w_-^+(t_o;\phi,\tau) \tag{D.40}$$

as claimed in Eq. (D.12).

(ii) The continuity of the map $t_o \mapsto \tilde{t}'_-(t_o;\phi,\tau)$ at $t_0 = 0$, hence on $[0,\ \bar{T}_o^-(\tau)]$, follows from Eqs. (D.39) and (D.40), to wit,

$$\lim_{t_o \to 0^+} \tilde{t}'_-(t_o;\phi,\tau) = \tau + \lim_{t_o \to 0^+} w_-^+(t_o;\phi,\tau) = \tau/C_1(\phi) = \tilde{t}'_-(0;\phi,\tau). \tag{D.41}$$

(iii) Since $\tilde{t}'(t_o;\ |\mathbf{x}|\hat{\mathbf{x}},\ |\mathbf{x}|/c + \tau) \rightarrow \tilde{t}'_-(t_o;\phi,\tau)$ t_0-pointwise on compact $[0,\ \bar{T}_o^-(\tau)]$ and $t_0 \mapsto \tilde{t}'_-(t_0;\phi,\tau)$ is continuous there, then by Dini's Theorem it is sufficient to show that the map $|\mathbf{x}| \mapsto \tilde{t}'(t_o;\ |\mathbf{x}|\hat{\mathbf{x}},\ |\mathbf{x}|/c + \tau)$ is monotonic for each $t_o \in [0,\ \bar{T}_o^-(\tau)]$; in fact, we show that there

exists $\mathcal{X} > 0$ such that $\forall_{t_o \in [0, \tilde{T}_o^-(\tau)]}\ |\mathbf{x}| > \mathcal{X} \rightarrow$ the aforementioned map is increasing. To that end, we apply -- as in Theorem C.4 -- the Implicit Function theorem to

$$\mathcal{H}(|\mathbf{x}|, w) \equiv w - |\mathbf{x}|/c + (|\mathbf{x}|/c)\{\sin^2\phi + [|\mathbf{x}|^{-1}Z(t_o; w + \tau) - \cos\phi]^2\}^{1/2} \tag{D.42}$$

to conclude *via* Eq. (D.14) that $|\mathbf{x}| \mapsto w(t_o; |\mathbf{x}|, \phi, \tau) = \tilde{t}'(t_o; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau) - \tau$ is C^1 at $\varkappa \in (0, \infty)$ whenever $\varkappa$ and $t_o \in [0, \tilde{T}_o^-(\tau)]$ satisfy

$$0 \neq \{\sin^2\phi + [\varkappa^{-1}Z(t_o; \tilde{t}'(t_o, \varkappa\hat{\mathbf{x}}, \varkappa/c + \tau)) - \cos\phi]^2\}^{1/2} \equiv R(t_o, \varkappa) \tag{D.43}$$

(where the ϕ and τ dependence of R have been supressed); and further, for such $\varkappa$ and t_o, that

$$(\partial w/\partial|\mathbf{x}|)(t_o; \varkappa, \phi, \tau) = (1/c)\{1 - R(t_o, \varkappa) + [R(t_o, \varkappa)]^{-1}[R_*(t_o, \varkappa) - \cos\phi]R_*(t_o, \varkappa)\}$$
$$\div \{1 + (1/c)[R(t_o, \varkappa)]^{-1}[R_*(t_o, \varkappa) - \cos\phi](\partial Z/\partial t)(t_o; \tilde{t}'(t_o; \varkappa\hat{\mathbf{x}}, \varkappa/c + \tau))\} \tag{D.44}$$

where

$$R_*(t_o, \varkappa) \equiv \varkappa^{-1}Z(t_o; \tilde{t}'(t_o, \varkappa\hat{\mathbf{x}}, \varkappa/c + \tau)) \tag{D.45}$$

and the denominator of $\partial w/\partial|\mathbf{x}|$ is strictly positive whenever the condition of Eq. (D.43) obtains since

$$(1/c)|[R_* - \cos\phi]/R|\cdot|\partial Z/\partial t| \leq 1/c \cdot 1 \cdot v_o = v_o/c < 1. \tag{D.46}$$

We now claim that there exists $\mathcal{X} > 0$ such that $\forall_{t_o \in [0, \bar{T}_o^-(\tau)]}$ $\varkappa \in (\mathcal{X}, \infty) \rightarrow R(t_o, \varkappa) > 0$ so that the condition of Eq. (D.43) obtains and the map $|\mathbf{x}| \mapsto w(t_o; |\mathbf{x}|, \phi, \tau)$ is C^1 on $(\mathcal{X}, \infty)$ and Eq. (D.44) is valid there; indeed, this positivity follows immediately from the definition of R in Eq. (D.43) by using Proposition D.2. Further, when $|\mathbf{x}| > \mathcal{X}$ then the numerator of $\partial w/\partial|\mathbf{x}|$ is positive, as follows. We first note that if $\cos\phi = 0$ then $R_*(t_o, |\mathbf{x}|)\cos\phi = 0 < 1$, while if $\cos\phi > 0$ then, again by Proposition D.2, there exists $\mathcal{X}' > 0$ such that $\forall_{t_o \in [0, \bar{T}_o^-(\tau)]}$ $|\mathbf{x}| > \mathcal{X}' \rightarrow R_*(t_o, |\mathbf{x}|)\cos\phi < 1$ and, WLOG, we may take $\mathcal{X}' = \mathcal{X}$. Thus in either case we have, since $\phi > 0$, that $R_*^2 > R_*^2\cos^2\phi$ so $\{R_*^2 - 2R_*\cos\phi + 1\}^{1/2} > 1 - R_*\cos\phi$; hence, noting from Eqs. (D.43) and (D.45) that

$$R = \{R_*^2 - 2R_*\cos\phi + 1\}^{1/2}, \tag{D.47}$$

we then have

$$R - R^{-1}[R_* - \cos\phi]R_* = R^{-1}[R^2 - (R_* - \cos\phi)R_*] = [1 - R_*\cos\phi]/R < 1 \tag{D.48}$$

so that the numerator of $\partial w/\partial|\mathbf{x}|$ is positive for all $t_o \in [0, \bar{T}_o^-(\tau)]$ whenever $|\mathbf{x}| > \mathcal{X}$. Thus if $|\mathbf{x}| > \mathcal{X}$ then $(\partial w/\partial|\mathbf{x}|)(t_o; |\mathbf{x}|, \phi, \tau) > 0$ for all $t_o \in [0, \bar{T}_o^-(\tau)]$ so the map $|\mathbf{x}| \mapsto \bar{t}'(t_o; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau)$ is increasing for such t_o.

(iv) This is straightforward to verify using Eq. (D.12). ■

Proposition D.4: Let $\hat{\mathbf{x}} = (\sin\phi, 0, \cos\phi)$ with $0 < \phi \le \pi/2$, $\tau > 0$, and $t_o \in [0, \bar{T}_o^-(\tau))$. Then

$$\lim_{|\mathbf{x}|\to\infty} V(t_o; \bar{t}'(t_o; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau)) = V(t_o; \bar{t}'_-(t_o; \phi, \tau)) \tag{D.49}$$

t_0-uniformly on $[0, \bar{T}_o^-(\tau)]$.

Proof. That the limit is as stated follows from Proposition D.3(i) and continuity of the map $t \mapsto V(t_o; t)$ at $t = \bar{t}'(t_o; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau)$, which continuity follows from Eq. (5.16) since $\tau > 0$. Further, since

$$V(t_o; \bar{t}'_-(t_o; \phi, \tau)) = v_o\{1 - 2T^{-1}G(\Omega t_o)[\bar{t}'_-(t_o; \phi, \tau) - t_o]\} \tag{D.50}$$

then, since $|G(\Omega t_o)| \le 1$ for all $t_o \in [0, \bar{T}_o^-(\tau)]$, we have for all such t_o

$$|V(t_o; \bar{t}'_-(t_o; \phi, \tau)) - V(t_o; \bar{t}'(t_o; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau))| \le 2T^{-1}|\bar{t}'_-(t_o; \phi, \tau) - \bar{t}'(t_o; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau)| \tag{D.51}$$

so by Proposition D.3(iii) the convergence in Eq. (D.49) is t_0-uniform on $[0, \bar{T}_o^-(\tau)]$. ■

It is easy to compute, using Eqs. (D.12) and (D.50), that, when $t_o \ne 0$,

$$V(t_o;\bar{t}'_-(t_o;\phi,\tau))/c = \begin{cases} (v_o/c)[1 - 2T^{-1}G(\Omega t_o)(\tau - t_o)], & \text{if } \phi = \pi/2 \\ (1/\cos\phi)\left(1 - C_1(\phi)\{1 + [2C_2(\phi)/C_1^2(\phi)]G(\Omega t_o)(\tau - t_o)\}^{1/2}\right), & \text{if } 0 < \phi < \dfrac{\pi}{2} \end{cases} \tag{D.52}$$

and of course

$$V(0;\bar{t}'_-(0;\phi,\tau))/c = v_o/c. \tag{D.53}$$

Since

$$[2C_2(\phi)/C_1^2(\phi)]G(\Omega t_o)(\tau - t_o) \le 4(v_o/c)T^{-1}G(\Omega t_o)T[G(\Omega t_o)]^{-1}/[1-(v_o/c)\cos\phi] < 4(v_o/c) < 1 \tag{D.54}$$

then the second line of Eq. (D.52) is given, correct to first order in v_o/c, by

$$V(t_o;\bar{t}'_-(t_o))/c \approx (v_o/c)[1 - 2T^{-1}G(\Omega t_o)(\tau - t_o)] \qquad (0 < \phi < \pi/2); \tag{D.55}$$

in fact, this last expression is exactly true for $\phi = \pi/2$, as Eq. (D.52) shows, and for $t_o = 0$ as well, as Eq. (D.53) shows. In Section V and beyond, we take Eq. (D.55) (Eq. (5.23)) to be our expression for $V(t_o;\bar{t}'_-(t_o;\phi,\tau))/c$ in all cases.

Proof of Theorem 5.1: By Proposition D.2, there exists $\mathcal{X} > 0$ such that

$\forall_{t_o \in [0, \bar{T}_o^-(\tau)]}\ |\mathbf{x}| \geq \mathcal{X} \rightarrow |\mathbf{x}|^{-1} Z(t_o; \bar{t}'(t_o; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau)) < 1$. Define functions k_1, k_2, and h as follows:

(a) $k_1 : [0, \bar{T}_o^-(\tau)] \times [\mathcal{X}, \infty) \rightarrow [-2/\sin\phi, 2/\sin\phi]$ by

$$k_1(t_o; |\mathbf{x}|) = \Lambda_o(t_o; \bar{t}'(t_o; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau))$$

$$= (1/\sin\phi)|\mathbf{x}|^{-1} Z(t_o; \bar{t}'(t_o; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau)) - \cot\phi$$

with $|k_1(t_o; |\mathbf{x}|)| < (1/\sin\phi) + \cot\phi < 2/\sin\phi$;

(b) $k_2 : [0, \bar{T}_o^-(\tau)] \times [\mathcal{X}, \infty) \rightarrow [-v_o/c, v_o/c]$ by

$$k_2(t_o; |\mathbf{x}|) = V(t_o; \bar{t}'(t_o; |\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau))/c;$$

(c) for $p, q \in \mathbb{Q}$, $p \geq 0$, $h_{p,q} : [0, \bar{T}_o^-(\tau)] \times [-2/\sin\phi, 2/\sin\phi] \rightarrow \mathbb{R}$ by

$$h_{p,q}(t_o, k) = G'(\Omega t_o) G(\Omega t_o) k^p (1 + k^2)^{-q}$$

with $|h_{p,q}(t_o, k)| \leq (2/\sin\phi)^p \left(\dfrac{4 + \sin^2\phi}{\sin\phi}\right)^{-q} < \infty$.

By Eq. (5.16), the maps $t_o \mapsto Z(t_o; \bar{t}')$ and $t_o \mapsto V(t_o; \bar{t}')$ are continuous at t_o, hence so are $k_i(t_o; |\mathbf{x}|)$, $i = 1,2$; further since G and G' are continuous on $[0, \bar{T}_o^-(\tau)]$, then so is h on its domain. Finally, by Propositions D.3 and D.4 and Eq. (D.55), $k_1(t_o; |\mathbf{x}|)$ converges uniformly to $-\cot\phi$ on

$[0, \bar{T}_o^-(\tau)]$ as $|\mathbf{x}| \rightarrow \infty$ and $k_2(t_o; |\mathbf{x}|)$ converges uniformly to $(v_o/c)[1 - 2T^{-1}G(\Omega t_o)(\tau - t_o)]$ (to first order in v_o/c) on $[0, \bar{T}_o^-(\tau)]$ as $|\mathbf{x}| \rightarrow \infty$. The results of this theorem then follow by applying Lemma D.1 to $\mathbf{E}_a(|\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau)$ and $\mathbf{B}_a(|\mathbf{x}|\hat{\mathbf{x}}, |\mathbf{x}|/c + \tau)$ as specified by Eqs. (5.1) - (5.3). ■